Fundamentals Circuit

Review for the Exit Exam

I-Basic Concepts

II-Basic Laws

III-(a) DC circuit analysis methods

-(b) Theorems

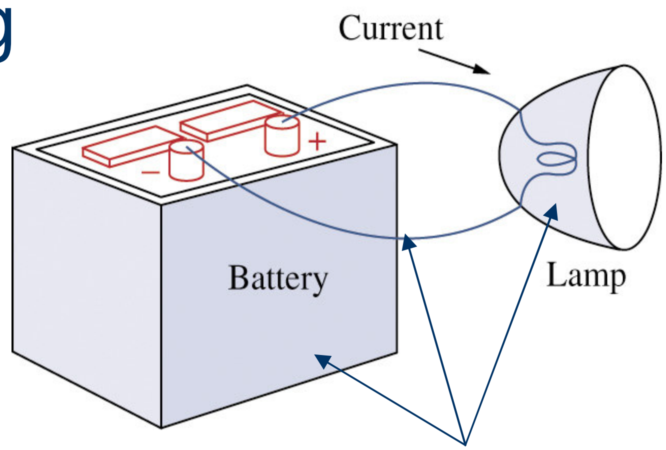
IV- Transient analysis

V-AC circuit analysis

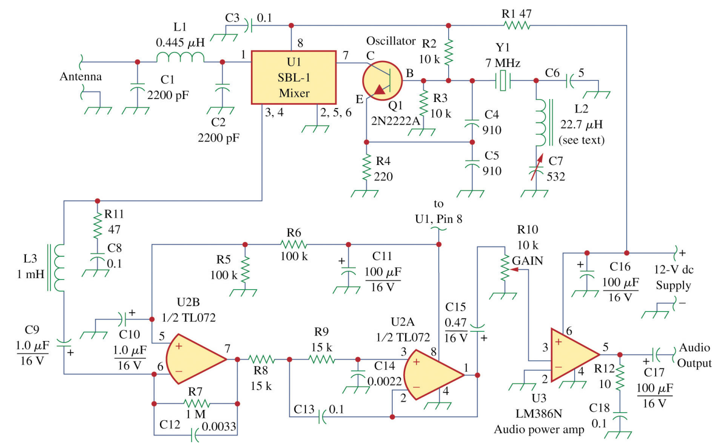
Electric Circuit

An *electric circuit* is an interconnection of electrical elements

e.g



elements



Simple electrical circuit

Complicated real circuit

Basic Concepts

- Charge
- Current
- Voltage
- Power
- Energy
- Circuit Elements

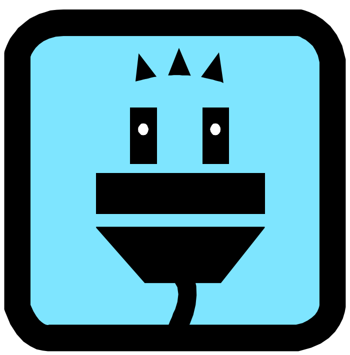
Charge

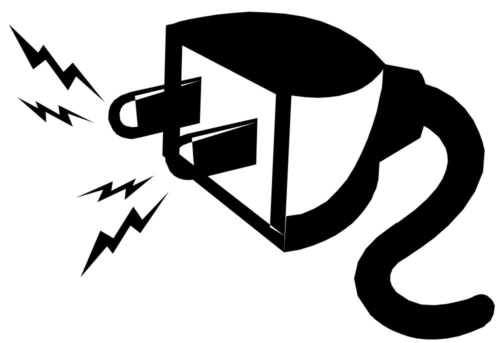
Charge is an electrical property of the atomic particles of which matter consists, measured in coulombs (C)

Note: An electron has a magnitude of 1.602 $X10^{-19}$C (with –ve sign)

1 electron = -1.602 $X10^{-19}$C

6.24 $X10^{18}$ electrons = -1 C

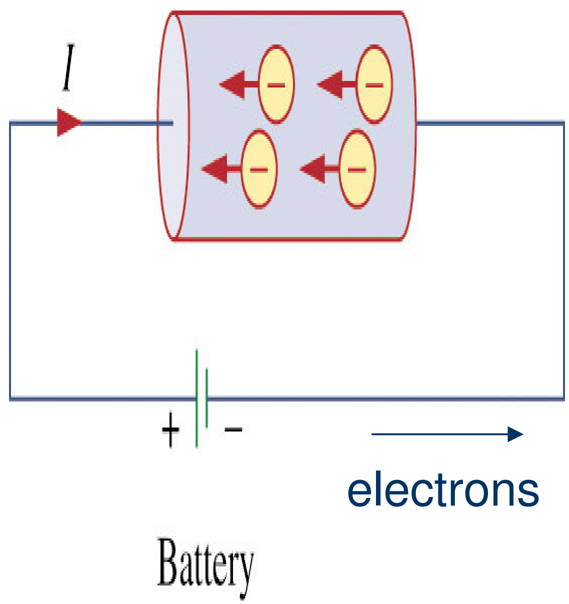




Flow of Electric Charge

Electric Charge is mobile, it can be transferred from one place to another.

When a conducting wire is connected to
a battery (a source of electromotive force),
the charges are compelled to move.
The movement of +ve and –ve charges
create electric current.



Note:

Current in a circuit is a flow of electrons. Electrons have a –ve charge. Because they are –ve the current is conventionally considered to be in the opposite direction to the flow of electrons

Current

Current is the time rate of charge, measured in amperes (A)

$$i = \frac{\mathrm{d}q}{\mathrm{d}t}$$

1 ampere = 1coulomb/second

$$Q = \int_{t_0}^{t} i \, \mathrm{d}t$$

Note:

If the current does not change with time, but remains constant, it is called direct current (dc) and the conventional symbol is I

An alternating current (ac) is a current that varies sinusoidally with time and the cpnventional symbol is i

Quiz 1

How much charge is represented by 4,600 electrons ?

1 electrons = $-1.602 \times 10^{-19}C$

4,600 electrons = $-7369.2 \times 10^{-19}C$

$(4{,}600 \times -1.602 \times 10^{-19}C)$

Voltage

Voltage (or potential difference) is the energy required to move a unit charge through an element, measured in volts (V).

$$v_{ab} = \frac{\mathrm{d}w}{\mathrm{d}q}$$

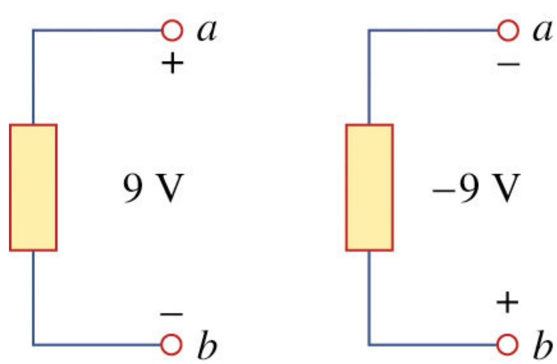
1 volt = 1 joule/coulomb

= 1 newton-meter/coulomb

Polarity of Voltage

Voltage across an element (represented by a rectangular block) connected to points a and b. The plus (+) and minus (-) signs are used to define reference direction or voltage polarity.

Points a is +9V above point b.
Or there is a 9-V voltage drop from a and b



(a) (b)

Points b is -9V above point a.
Or 9-V voltage rise from b to a

Note:
A constant voltage is called a *dc* voltage and represented by V
A sinusoidally time-varying voltage is called an *ac* voltage and is represented by v

Power

Power is the time rate of expending or absorbing energy, measured in watts (W).

$$p = \frac{\mathrm{d}w}{\mathrm{d}t}$$

1 watt = 1 joule/second

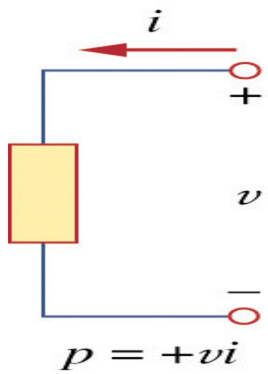
$$p = \frac{\mathrm{d}w}{\mathrm{d}q} \cdot \frac{\mathrm{d}q}{\mathrm{d}t}$$

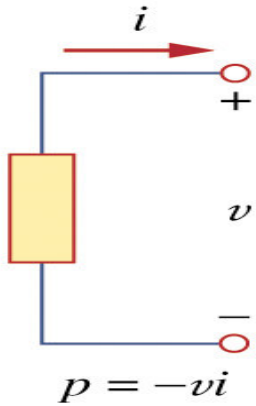
$$p = vi$$

To relate power and energy to voltage and current

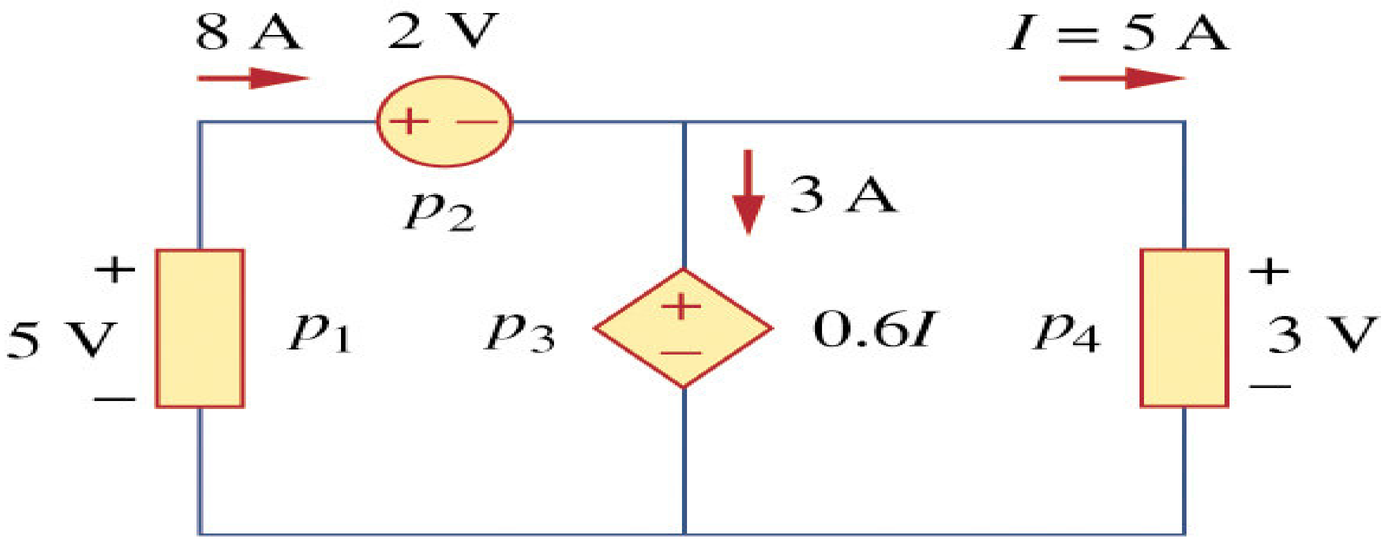
Passive sign convention

- By convention, the power of an element is:
  - **The positive product of voltage and current** if the element is absorbing power
    - i.e. current enters the positive terminal
  - **The negative product of voltage and current** if the element is supplying power
    - i.e. current enters the negative terminal
- This allows **conservation of energy**
  - the algebraic sum of power in a circuit is zero
  - i.e. [supplying power (-)] + [absorbing power (+)]=0





Quiz 2



$P_1 = 5(-8) = -40W$

$P2 = 2(8) = 16W$

$P3 = 0.6(5)(3) = 9W$

$P4 = 3(5) = 15W$

Energy

Energy is capacity to do work, measured in joules (J).

Note:The electric power utility companies measure energy in watt-hours (Wh), where

$$1\text{Wh} = 3{,}600 \text{ J}$$

Quiz 3

A stove element draws 15A when connected to a 120V line. How long does it takes to consume 30kJ?

$i = 15\text{A}$
$V = 120\text{V}$
$P = VI$
$= 15 \times 120$
$= 1800\ \text{W}$

$\text{P} = \text{d}w / \text{d}t$
$t = w/p$
$= 30000/1800$
$= 16.7s$

Circuit Elements

An element is the basic building block of a circuit.

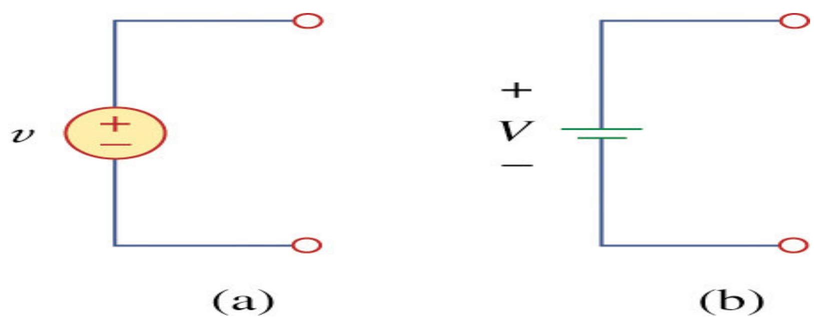
Type of elements:

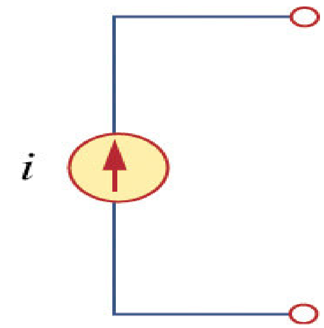
Passive Elements	Active Elements
Not capable of generating energy	*Capable of generating energy*
●Resistors	●Generators
●Capacitors	●Batteries
●inductors	●Operational amplifier

Sources

There are two kinds of sources: Independent and dependent.

An *ideal independent* source is an active element that provides a specified voltage or current that is completely independent of other circuit elements.





Symbols for independent voltages sources:

a)Constant or time-varying voltage

b)Constant voltage (dc)

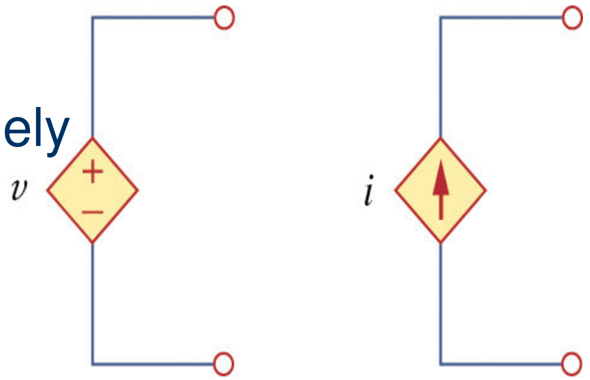
Symbols for independent current sources:

An *ideal dependent* (or controlled) source is an active element in which the source quantity is controlled by another voltage or current.

Dependent sources are usually designated by diamond-shaped symbols.

There are 4 types of dependent sources, namely

1. A voltage-controlled voltage source (VCVS)
2. A current-controlled voltage source (CCVS)
3. A voltage-controlled current source (VCCS)
4. A current-controlled current source (CCCS)



(a) (b)

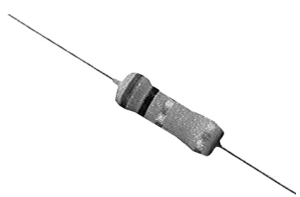
II-Basic Laws

- Ohm's Law
- Kirchhoff's Laws
  - Kirchhoff's Current Law (KCL)
  - Kirchhoff's Voltage Law

Resistance (R)

Resistance (*R*) of an element denotes its ability to resist the flow of electric current, it is measured in ohms (Ω.)

Circuit element used to model current-resisting behaviour of a material is the resistor.



Carbon film type

Wirewound type

Ohm's Laws

Ohms' law states that the voltage *v* across a resistor is directly proportional to the current *i* flowing through the resistor.

The constant of proportionality is termed the resistance, *R:*

$$v = iR$$

Two extreme possible values of *R*

Short Circuit is a circuit with resistance approaching zero, $R = 0$

Open Circuit is a circuit element with resistance approaching infinity, $R = \infty$

Conductance

The inverse of Resistance is Conductance

Conductance is the ability of an element to conduct electric current; it is measured in mhos (℧) or Siemens (S)

Network Topology

In network topology, we study the properties relating to the placement of elements in the network and the geometric configuration of the network. Such elements include:

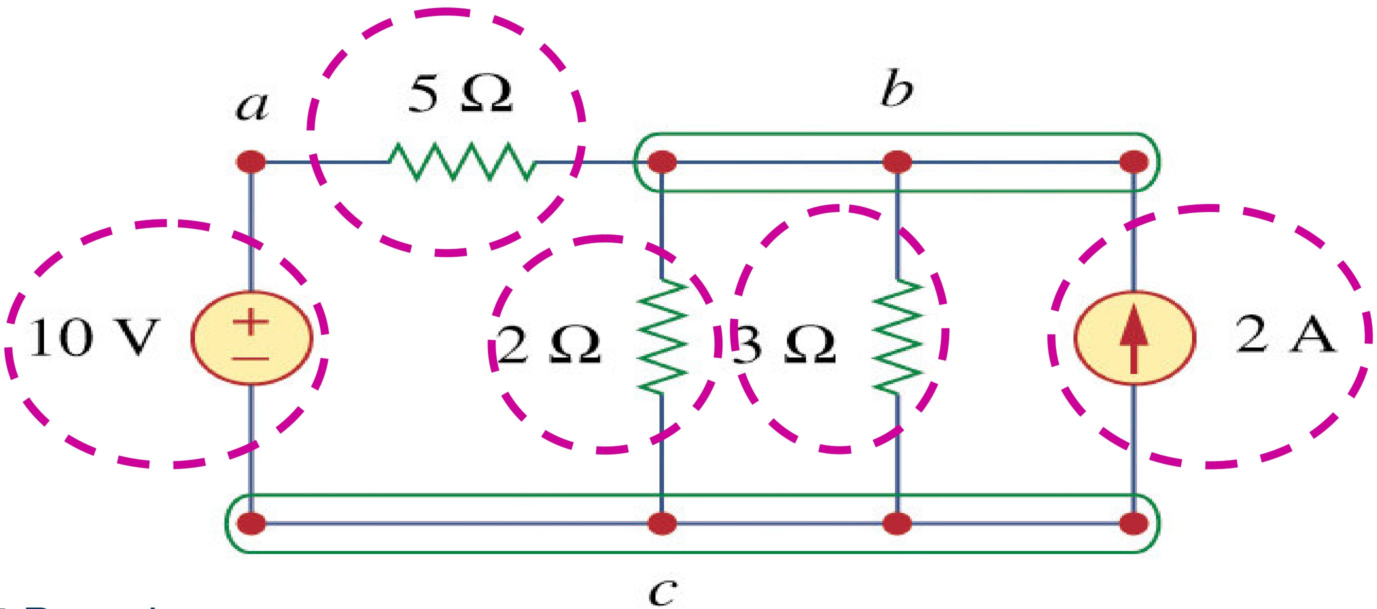
i) Branches

ii) Nodes

iii) Loops.

Branch

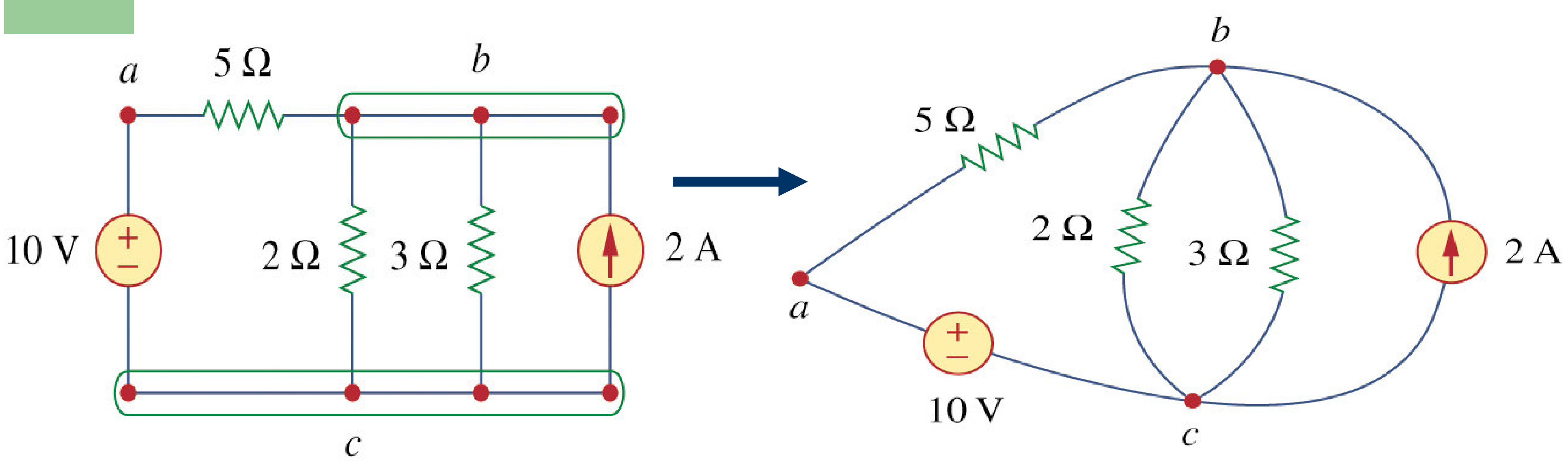
Branch represents a single element such as a voltage source or a resistor



5 Branches

Nodes

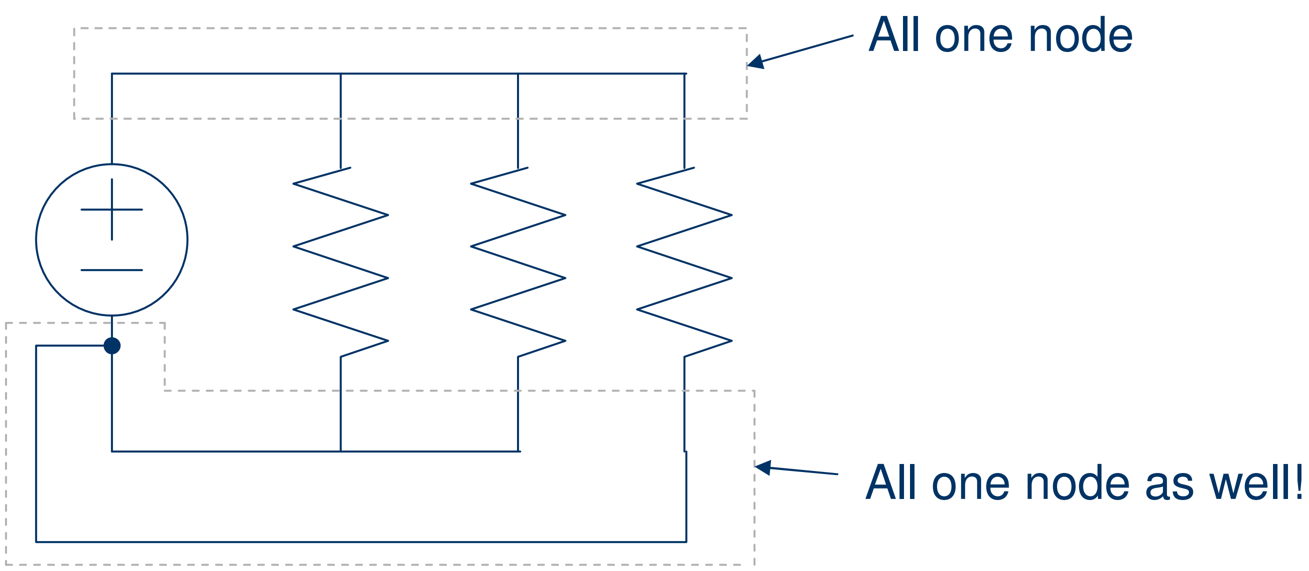
Nodes are the point of connection between two or more branches



5 branches → 3 nodes

Nodes 2

Recognising nodes is very important:

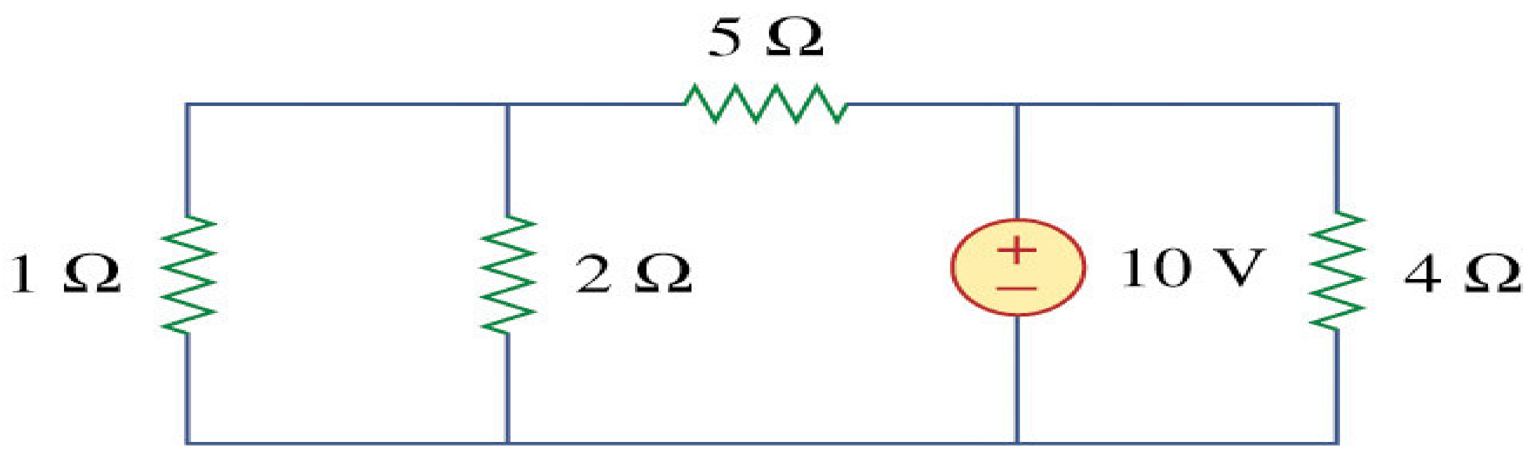


Loop

- A loop is a closed path through the circuit
- It may not pass the same node twice
- If it has at least one branch not in any other independent loop, it is an independent loop
- Branches = Loops(indep) + Nodes – 1
- If two elements share one node → series
- If two elements share two nodes → parallel

Quiz 1

How many branches and nodes does the circuit in the below figure have? Identify the elements that are in series and in parallel.

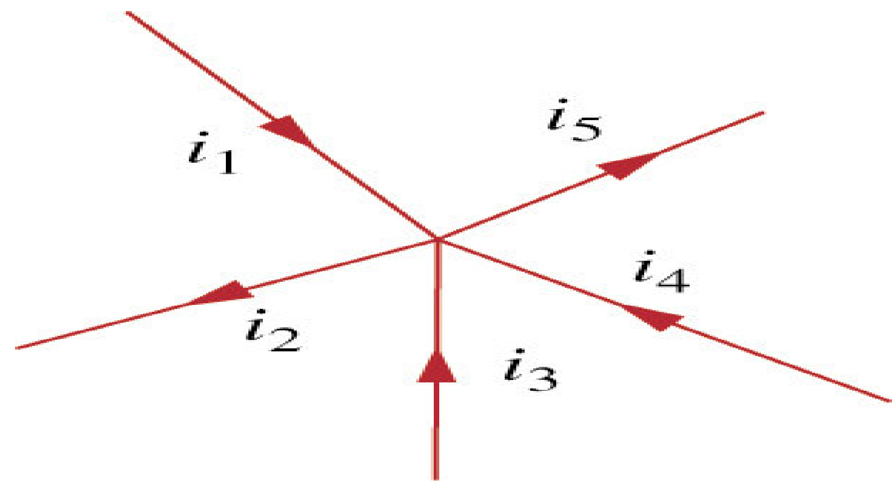


5 branches, 3 nodes, 1Ω, 2Ω are in parallel. 10V and 4Ω are in parallel

Kirchhoff's Current Law (KCL)

KCL states that the algebraic sum of currents entering a node (or a closed boundary) is zero

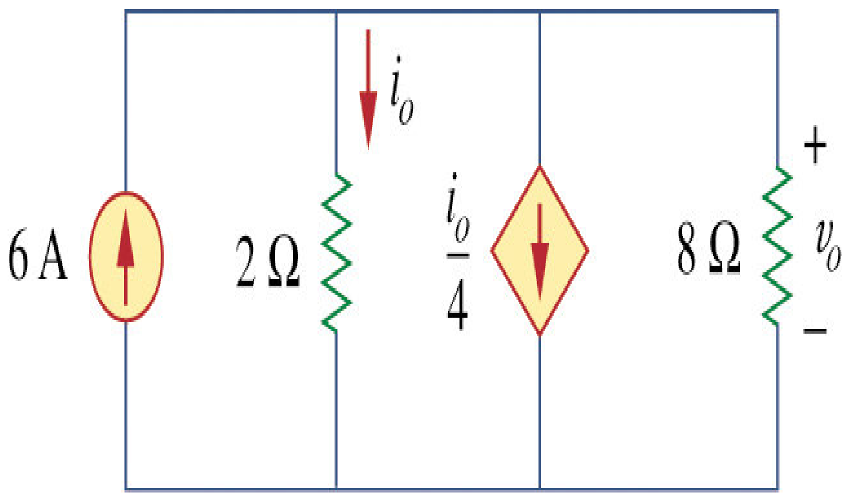
Alternative Form of KCL
The sum of the currents entering a node is equal to the sum of the currents leaving the node



$$i_1 + i_3 + i_4 = i_2 + i_5$$

Quiz 2

Find v_o and i_o in the circuit



$$-6+i_0+\frac{i_0}{4}+\frac{v_0}{8}=0$$

$$v_0=i_0(2)$$

$$-6+i_0+\frac{i_0}{4}+\frac{i_0}{4}=0$$

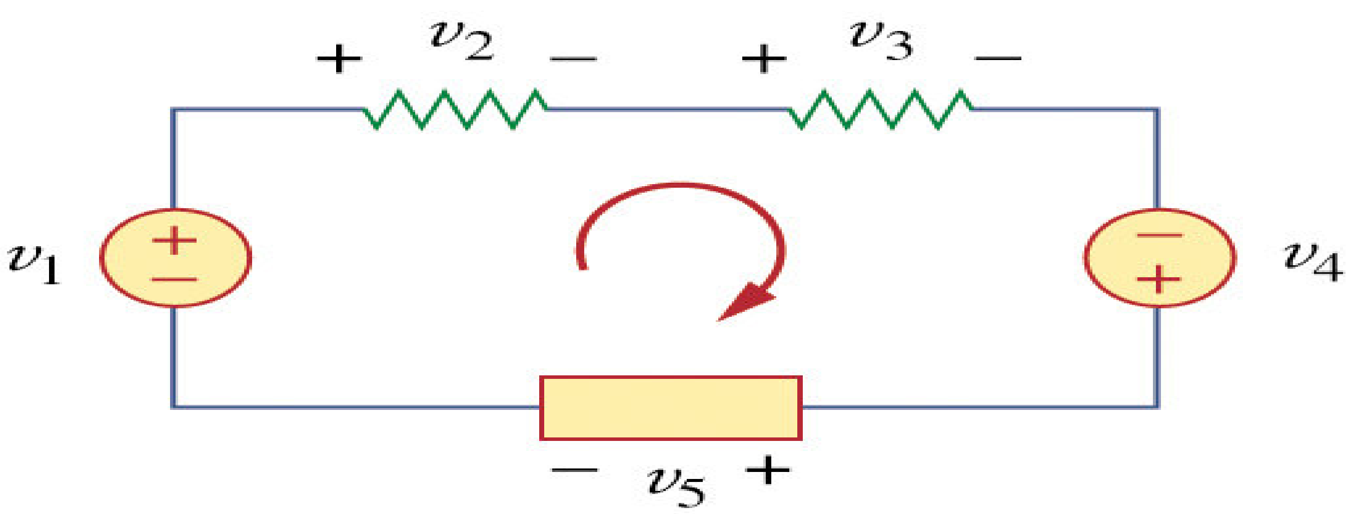
$$-24=-6i_0$$

$$i_0=4\text{A}$$

$$v_0=4(2)=8\text{V}$$

Kirchhoff's Voltage Law (KVL)

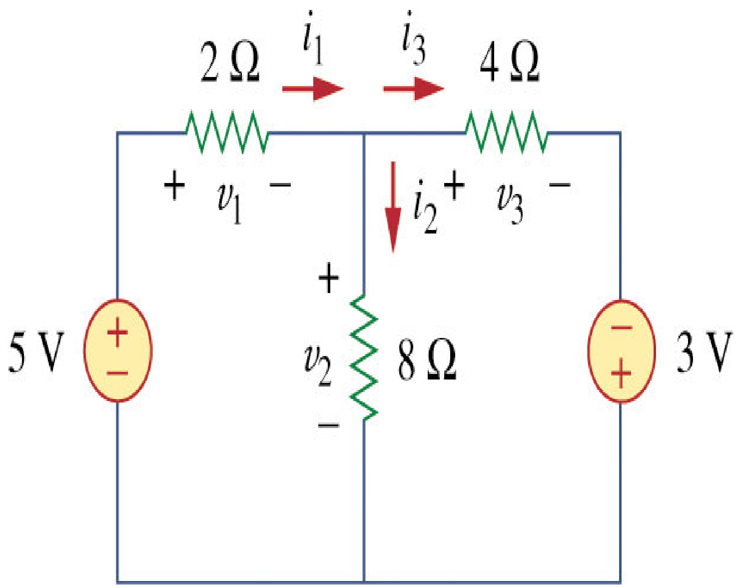
KVL states that the algebraic sum of all voltages around a closed path (or loop) is zero



$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$

Quiz 3

Find the currents and voltages in the circuit



KVL

$$-5 + 2i_1 + 8i_2 = 0$$

$$-8i_2 + 4i_3 - 3 = 0$$

KCL

$$i_1 - i_2 - i_3 = 0$$

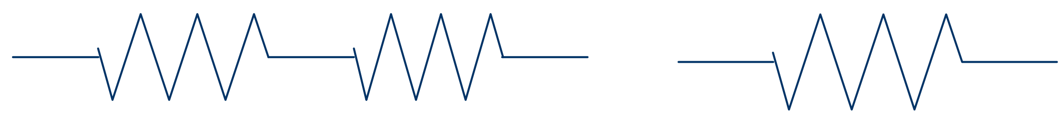
v_1 = 3V, v_2=2V, v_3= 5V,

i_1=1.5A, i_2=0.25A, i_3=1.25A

Equivalent Resistance

- The equivalent resistance is the resistance of an element which could replace all the resistors in the loop.
- For resistors in series, just add each resistance together to get the equivalent
- For resistors in parallel, add each conductance together to get the equivalent conductance, and then find the resistance from there

Series Resistor



Summation:

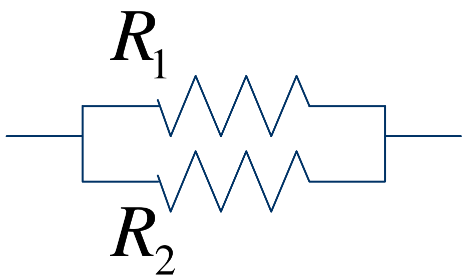
$$R_1 + R_2 = R_T$$

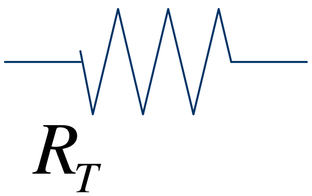
By Ohm's Law:

$$\frac{V_1}{I} + \frac{V_2}{I} = \frac{V_T}{I}$$

The above is used to calculate voltage division in series

Parallel Resistor



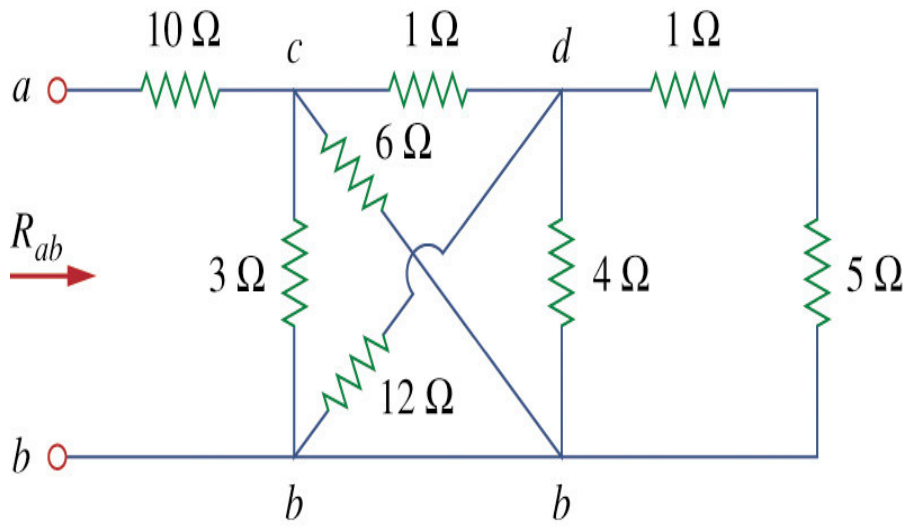


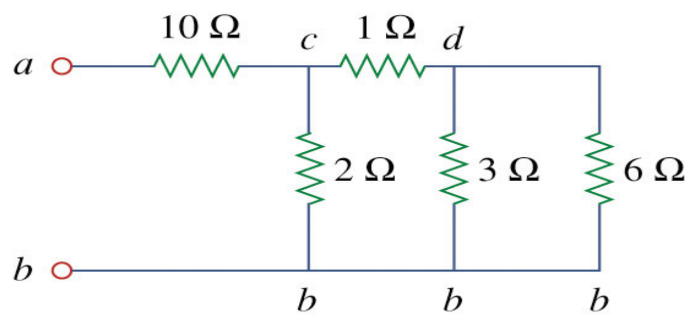
Summation of Conductance:

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_T}$$

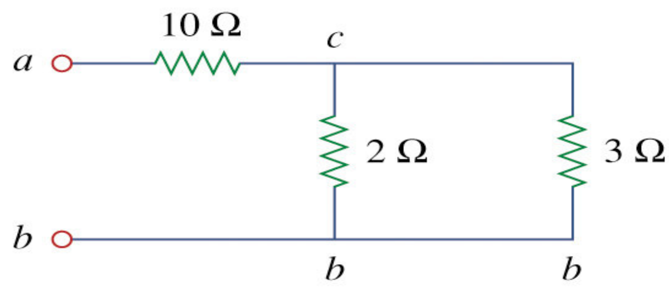
$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$

Quiz 4





(a)



(b)

3-Ω and 6-Ω are parallel

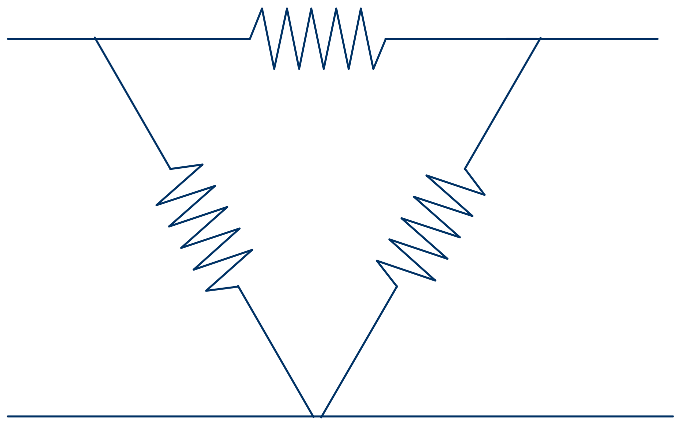
3x6/3+6 = 2-Ω

12-Ω and 4-Ω are parallel

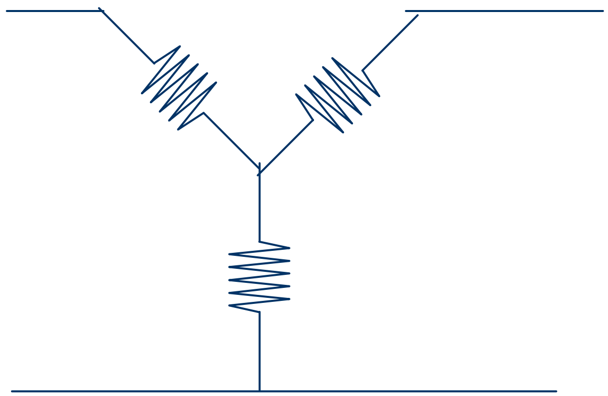
12x4/12+4=3-Ω

Delta to Wye

- Some forms of circuit are hard to analyse



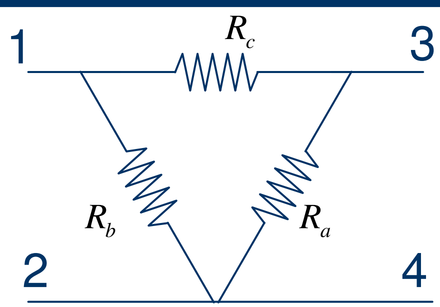
‘Delta’ configuration
Hard to analyse

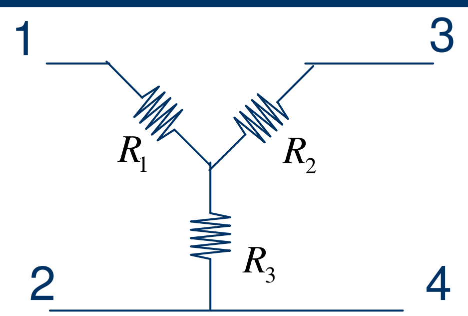


‘Y’ / ‘Wye’ configuration
Easier to analyse

<u>OBJECTIVE: Change ‘Delta’ configurations to ‘Y’ configurations</u>

Delta to Wye 2





1) Write the equivalent equations for each resistor in Delta in terms of Wye

$$R_{12}(Y) = R_1 + R_3$$

$$R_{12}(\Delta) = \frac{1}{\dfrac{1}{R_b} + \dfrac{1}{R_a + R_c}} = \frac{R_b(R_a + R_c)}{R_a + R_b + R_c}$$

$$R_{13}(Y) = R_1 + R_2$$

$$R_{13}(\Delta) = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c}$$

$$R_{34}(Y) = R_2 + R_3$$

$$R_{34}(\Delta) = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c}$$

Delta to Wye 3

Rearranging the previous equations produces:

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} \qquad R_2 = \frac{R_c R_a}{R_a + R_b + R_c} \qquad R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

Note that this is of the form:

Resistance of Y-resistor = product of two adjacent Delta-resistors divided by sum of Delta-resistances

- The method of the Delta to Wye conversion is more useful than the resulting equations
- This can be applied in many difficult to analyse situations, practice Wye to Delta as well

Wye to Delta

- The reverse procedure to the Delta to Wye conversion produces a Wye to Delta conversion.
- The result is left as an exercise (answer below)

Answer:
Each resistor in the Delta network is the sum of all possible products of Y taken two at a time, divided by the opposite Y resistor.

Wye to Delta (2)

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$

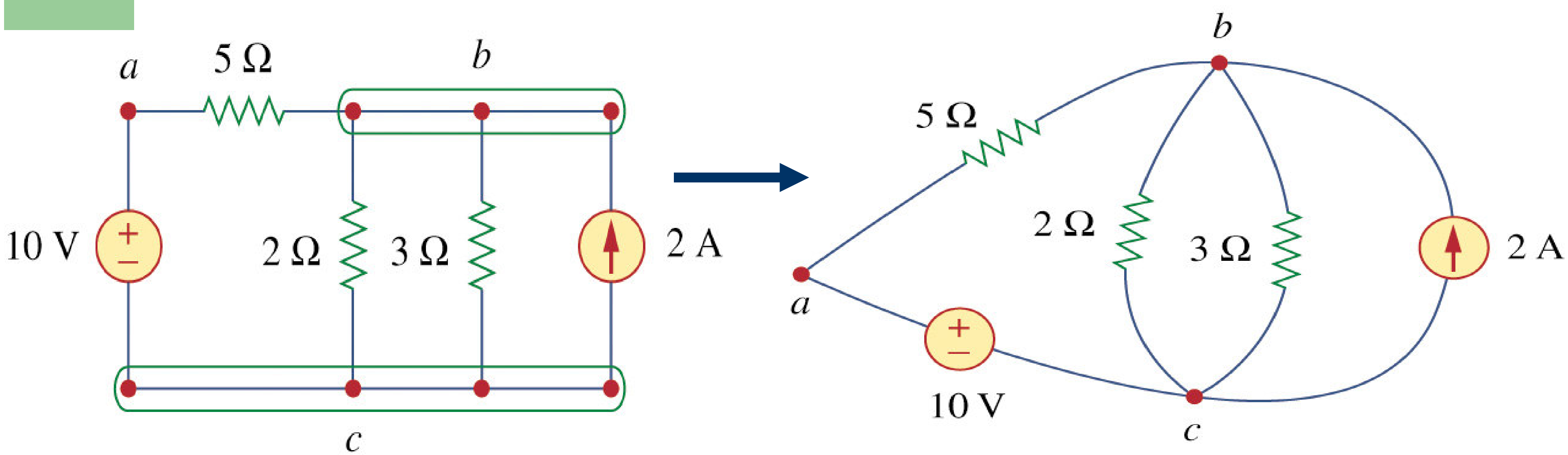
$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

III-Methods of Analysis

Reminder

Nodes are the point of connection between two or more branches



5 branches → 3 nodes

Techniques for Circuit Analysis

- *Nodal Analysis*

  Nodal analysis provides a general procedure for analyzing circuits using node voltages as the circuit variables.

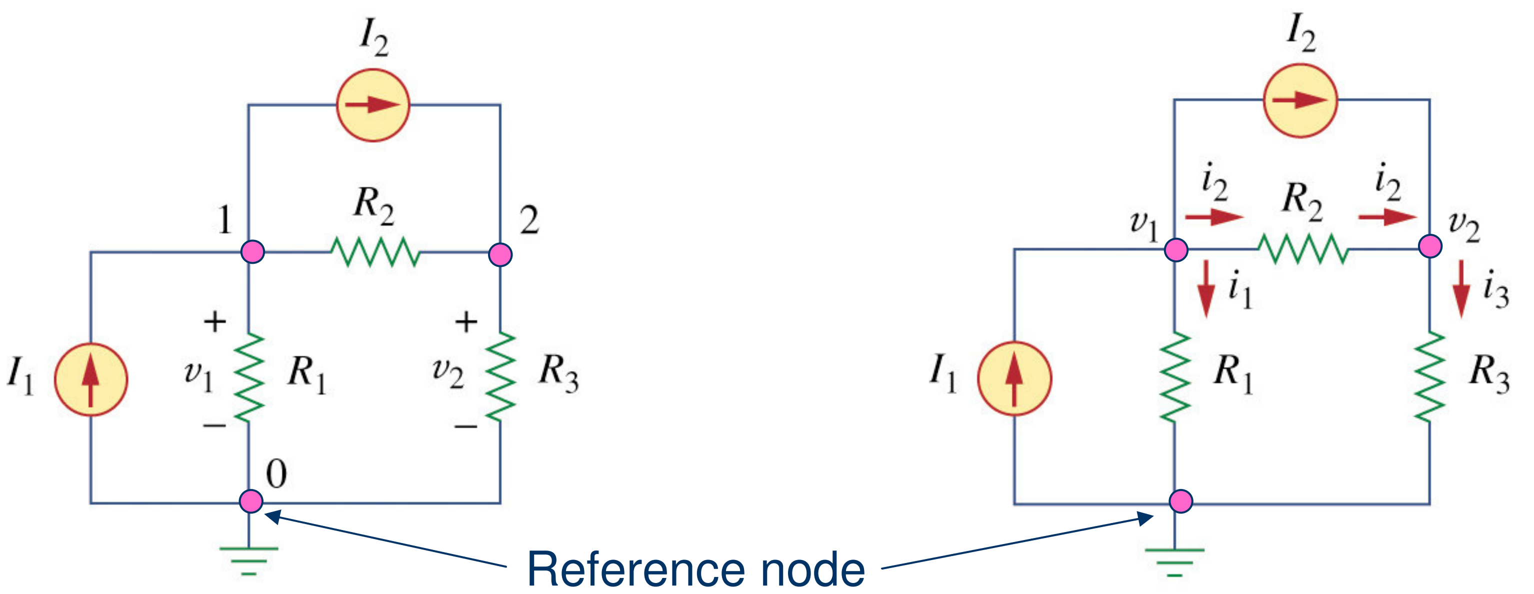
- *Mesh Analysis*

  Mesh analysis provides a general procedure for analyzing circuits using mesh currents as the circuit variables.

Determining Nodal Voltages Without Voltage Source (Nodal Analysis)

- Choose a node as a reference node
- Number all nodes based on the reference node (v_1 v_2 v_3 etc.)
- Apply KCL to each of the non-reference nodes
- Solve the resulting equations

Nodal Voltage Example

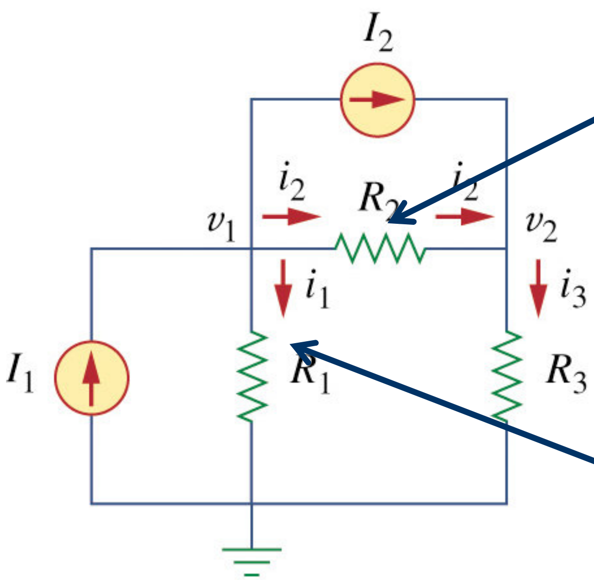


(a) (b)

The nodes are numbered 1, 2…
Number all nodes based on the reference node (v1 v2 v3 etc.)

Write the current on each branch numbered according to the node.

Nodal Voltage Example



(b)

$$i_2 = \frac{v_1 - v_2}{R_2}$$

$$i_3 = \frac{v_2 - 0}{R_3}$$

$$i_1 = \frac{v_1 - 0}{R_3}$$

Write Ohm’s law for each branch between node 1 and another node

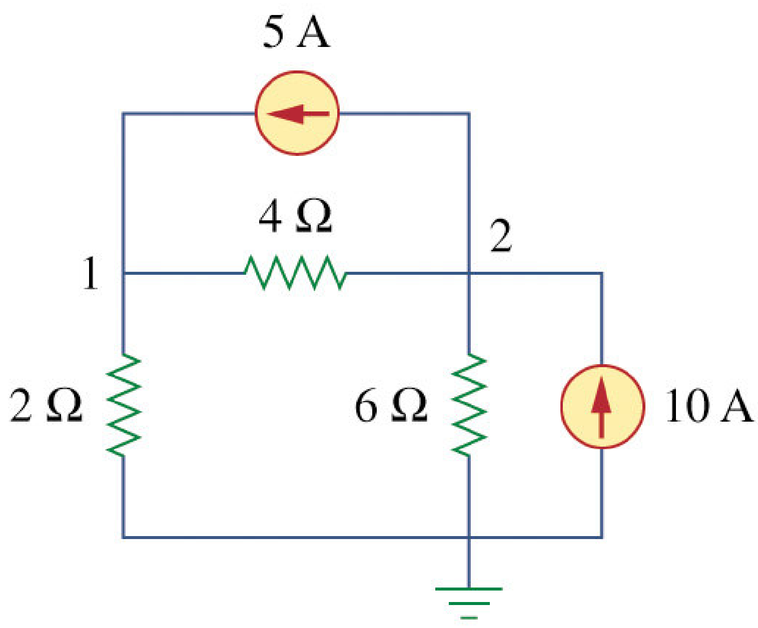
Now… solve these simultaneous equations with the data available

Nodal Voltage Example (2)

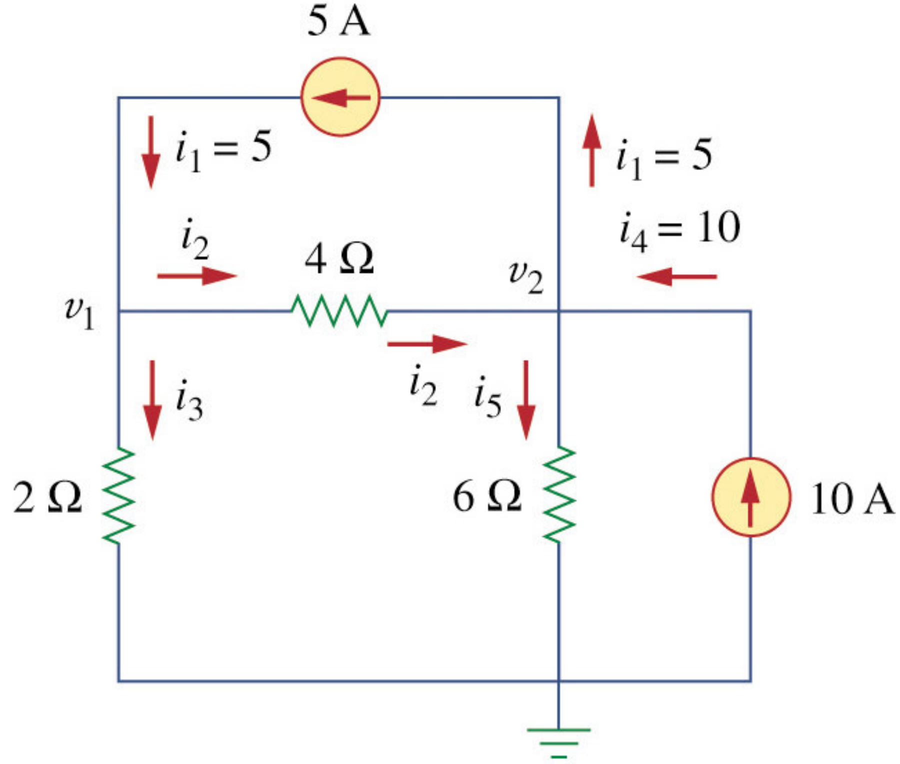
$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

Example

Calculate the node voltages in the circuit

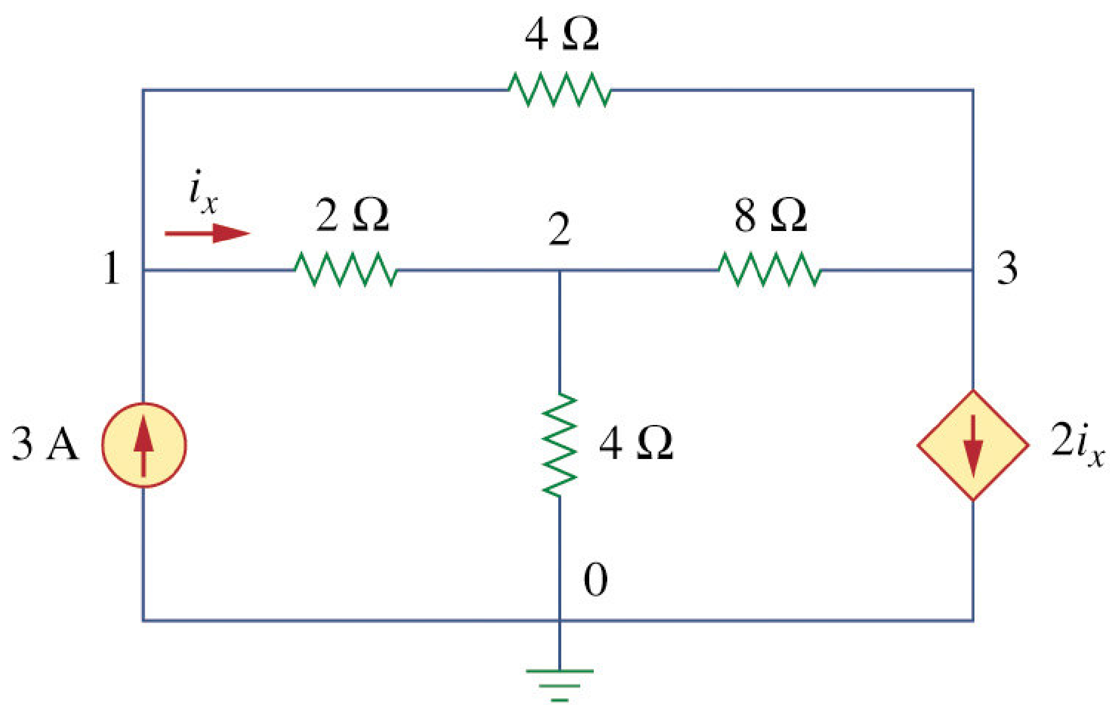


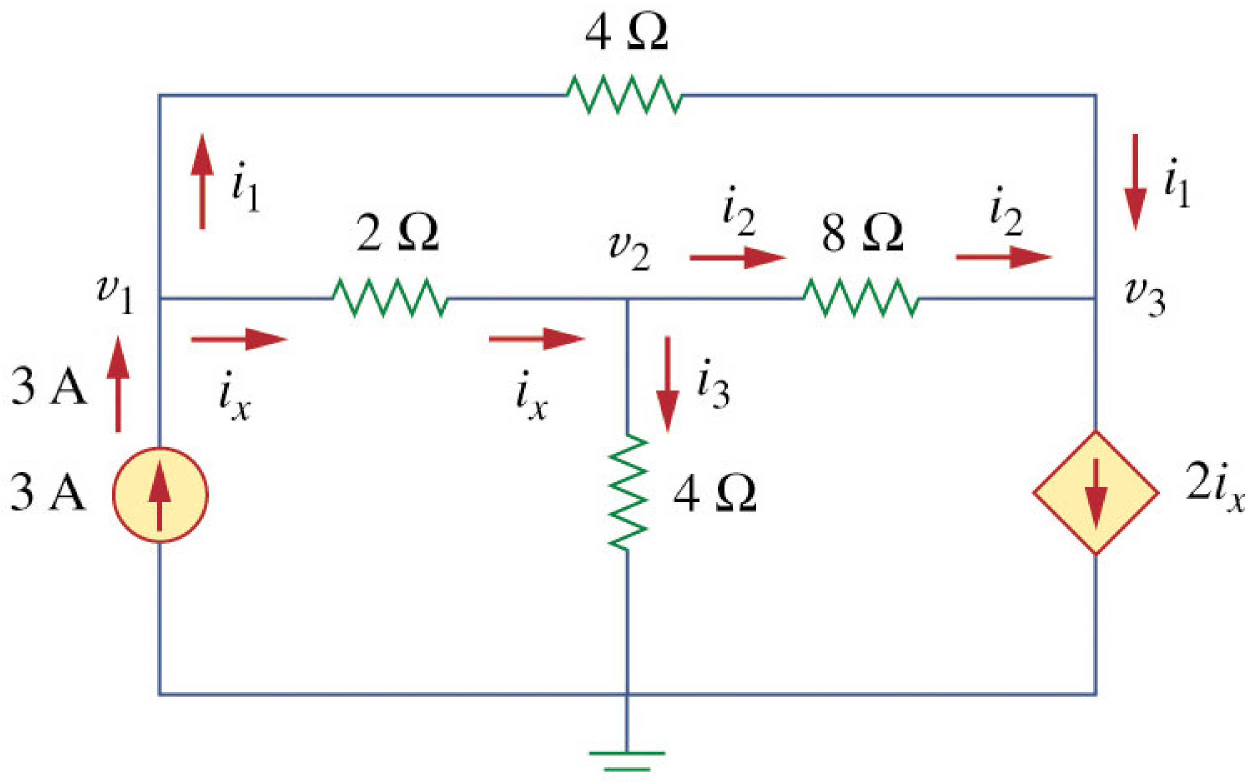
Circuit For Analysis



Example

Determine the voltages at the nodes



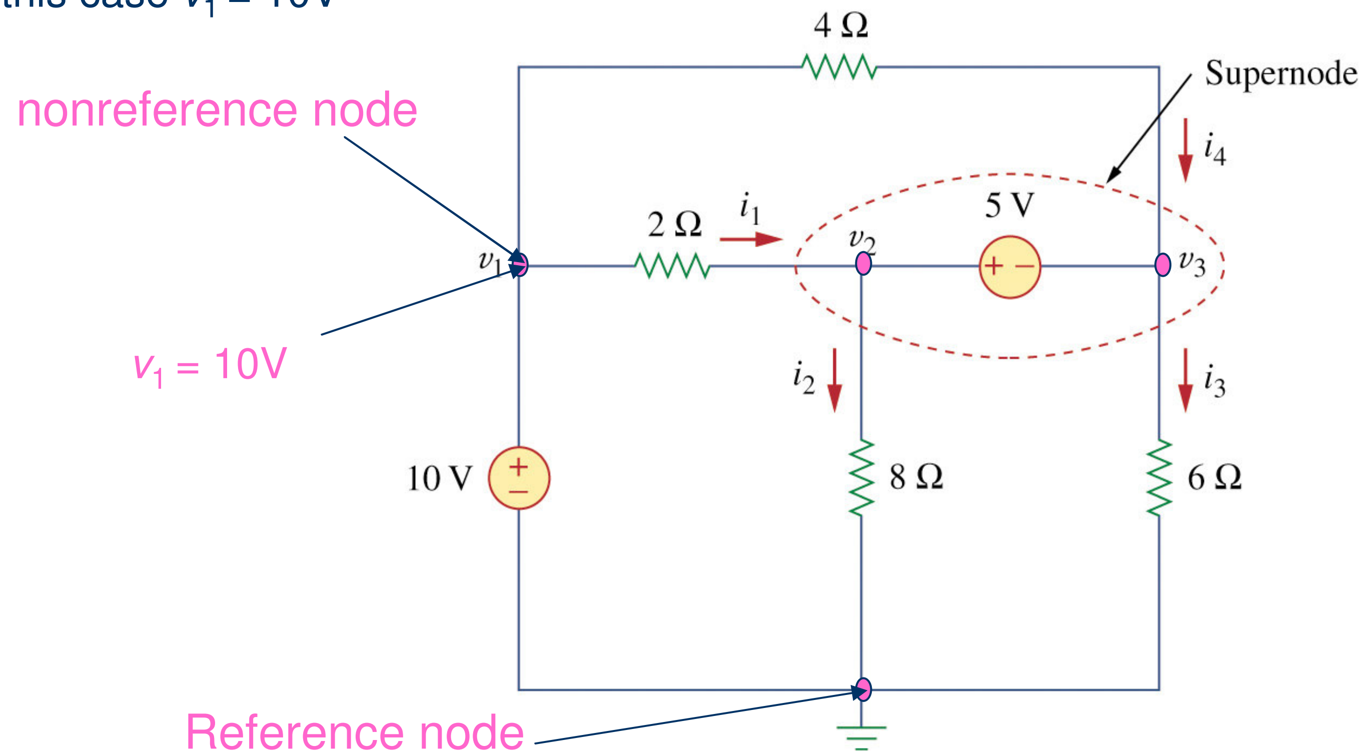


4 Ω
i_1
i_1
2 Ω
v_2
i_2
8 Ω
i_2
v_1
v_3
3 A
i_x
i_x
i_3
3 A
4 Ω
$2i_x$

Determining Nodal Voltages With Voltage Source (Nodal Analysis)

Situation 1

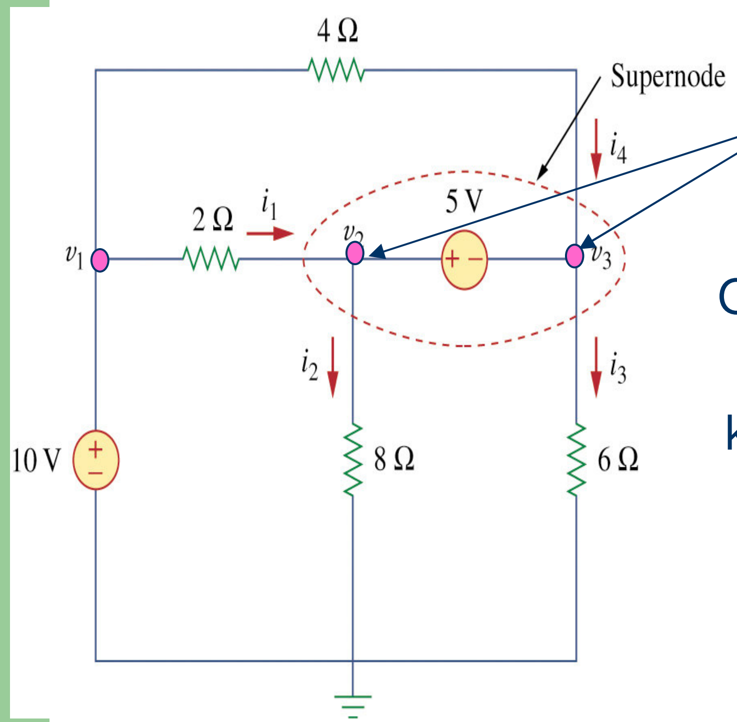
If a voltage source is connected between the reference note and non reference note, the voltage at the nonreference node equal to the voltage of voltage source, in this case v_1 = 10V



Determining Nodal Voltages With Voltage Source (Nodal Analysis)

Situation 2

If a voltage source (dependent or independent) is connected between two nonreference nodes, the two nonreference nodes form a generalized node or supernode; both KCL and KVL can be applied to determine the node voltages.



Nonreference Nodes

KCL must be satisfied at a supernode, hence

$$i_1 + i_4 = i_2 + i_3$$

Or

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6}$$

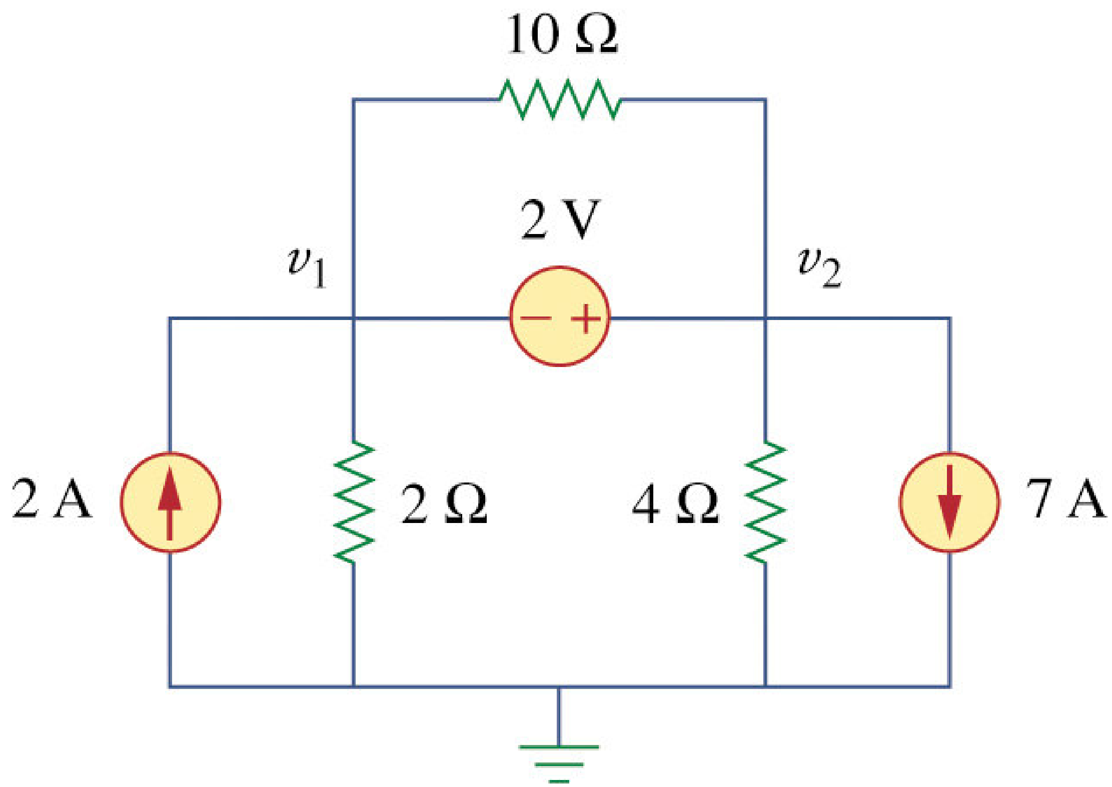
known

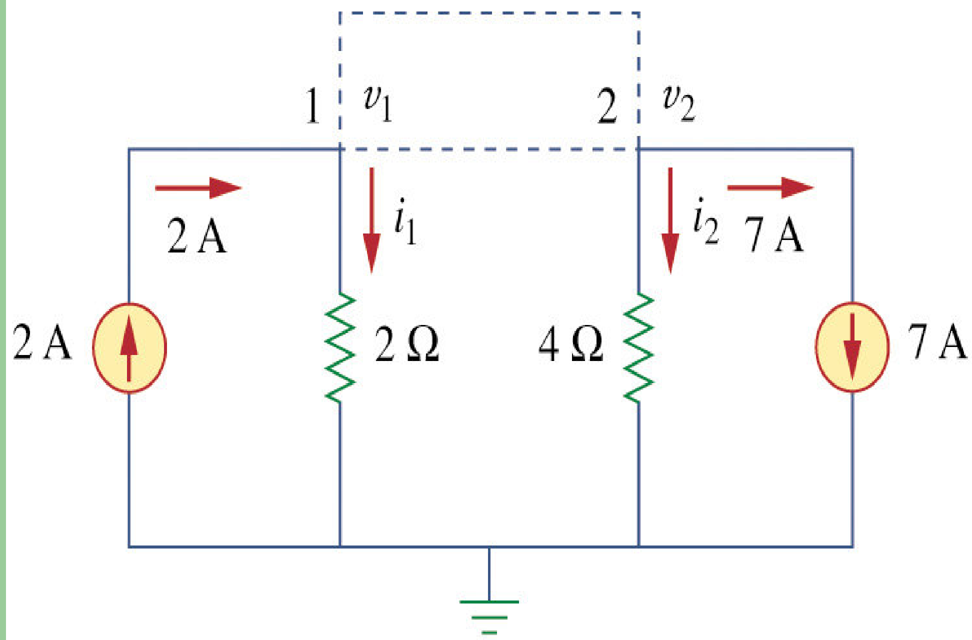
$$v_1 = 10$$

$$v_2 - v_3 = 5$$

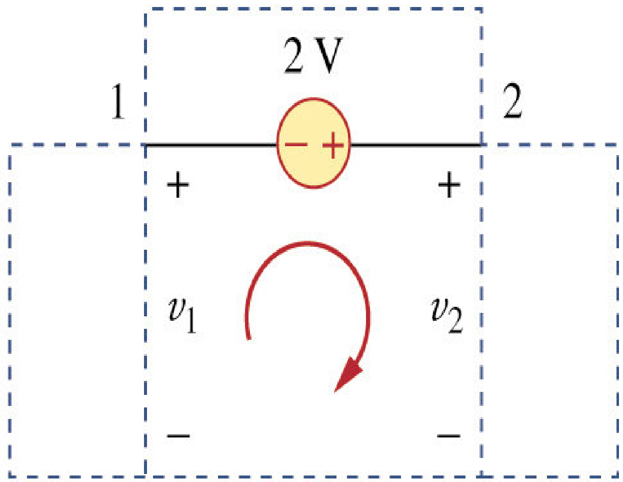
Example

For the circuit shown in figure, find the node voltage





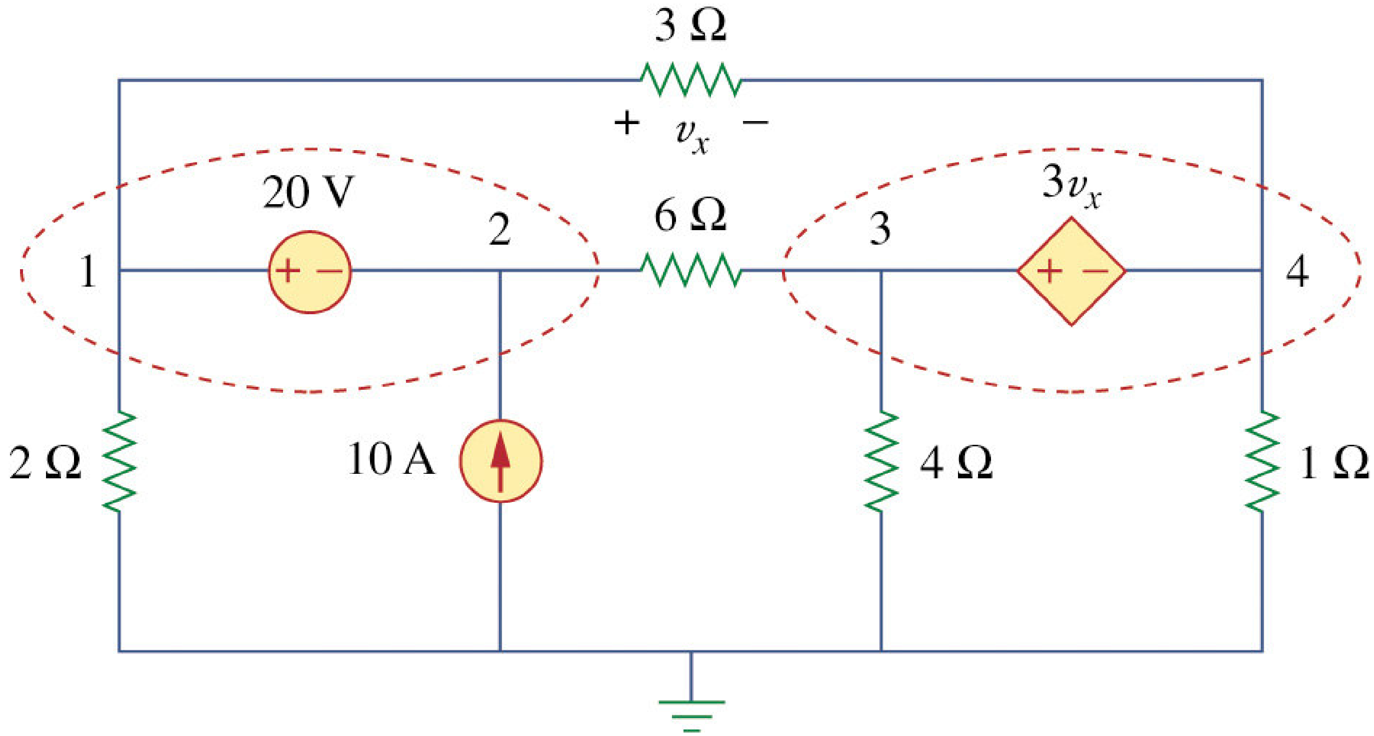
(a)

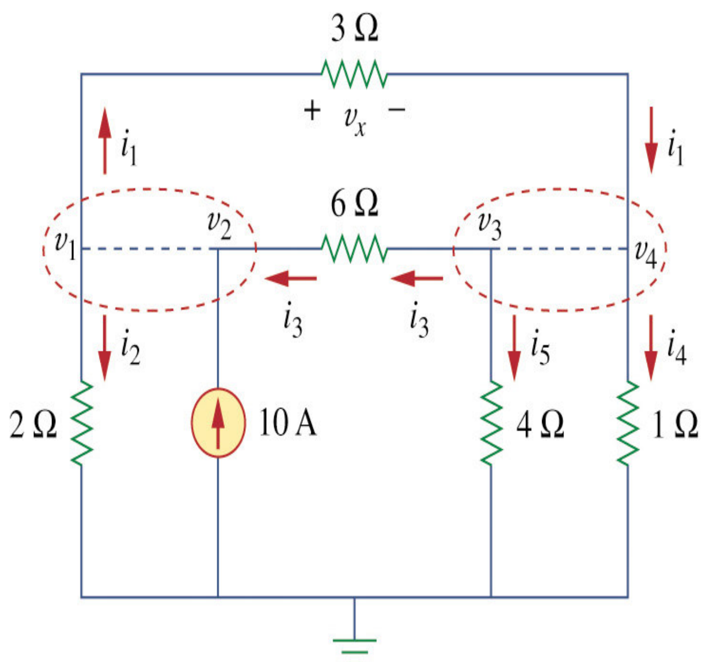


(b)

Example

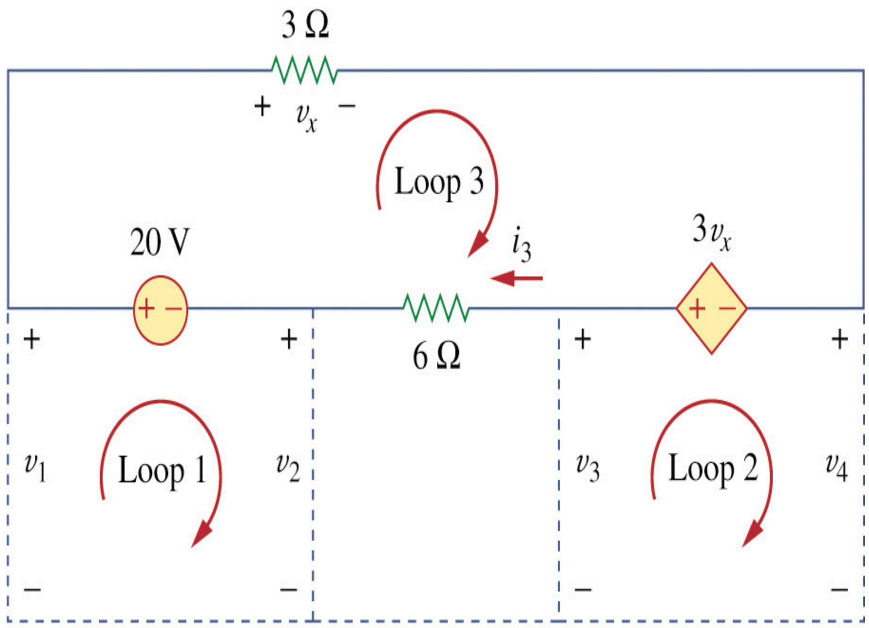
Find the node voltage in the circuit





3 Ω
+ v_x −
i_1
i_1
6 Ω
v_1
v_2
v_3
v_4
i_3
i_3
i_2
i_5
i_4
2 Ω
10 A
4 Ω
1 Ω

(a)

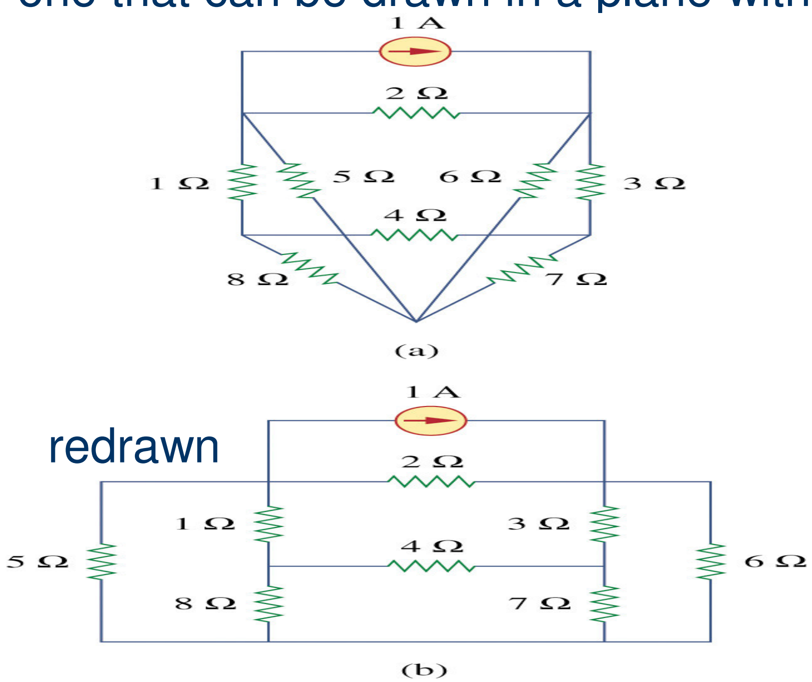


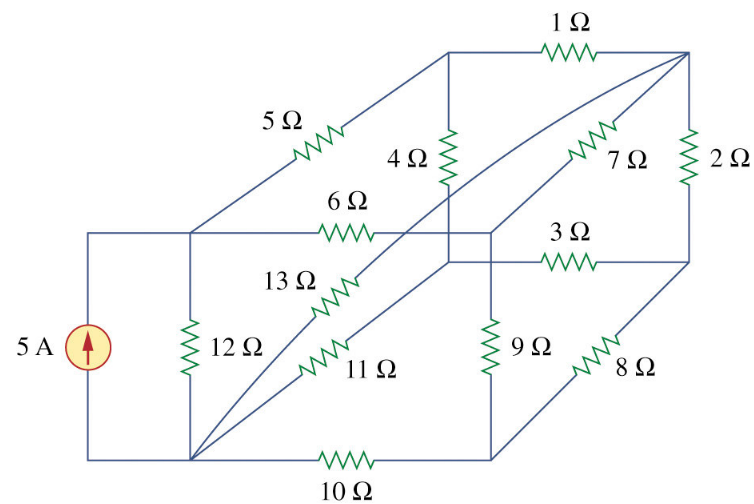
3 Ω
+ v_x −
Loop 3
20 V
i_3
$3v_x$
6 Ω
v_1
Loop 1
v_2
v_3
Loop 2
v_4

(b)

Mesh Analysis

Mesh analysis is only applicable to a circuit that is planar. A planar circuit is one that can be drawn in a plane with no branches crossing one another.





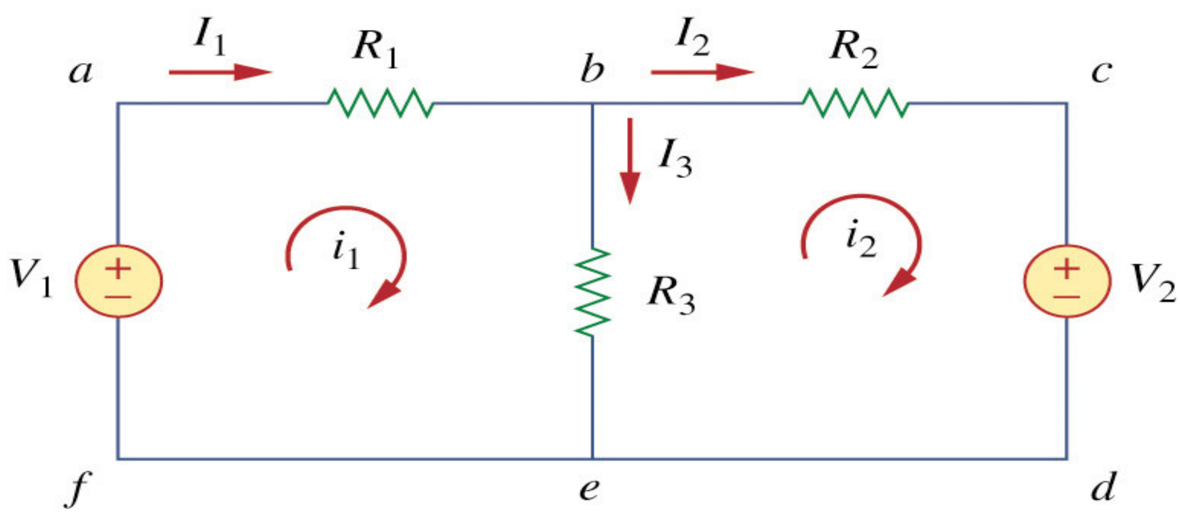
A planar circuit

A nonplanar circuit

Note: A mesh loop is a loop which does not contain any other loops within it

What is Mesh?

A mesh is a loop which does not contain any other loops within it



Paths abefa and bcdeb are meshes. The current through a mesh is known as mesh current

Determine Mesh Currents Without Current Source (Mesh Analysis)

- Assign mesh currents $i_1, i_2, \ldots, i_n$ to the n meshes
- Apply KVL to each of the n meshes
- Solve the resulting n simultaneous equations to get the mesh currents

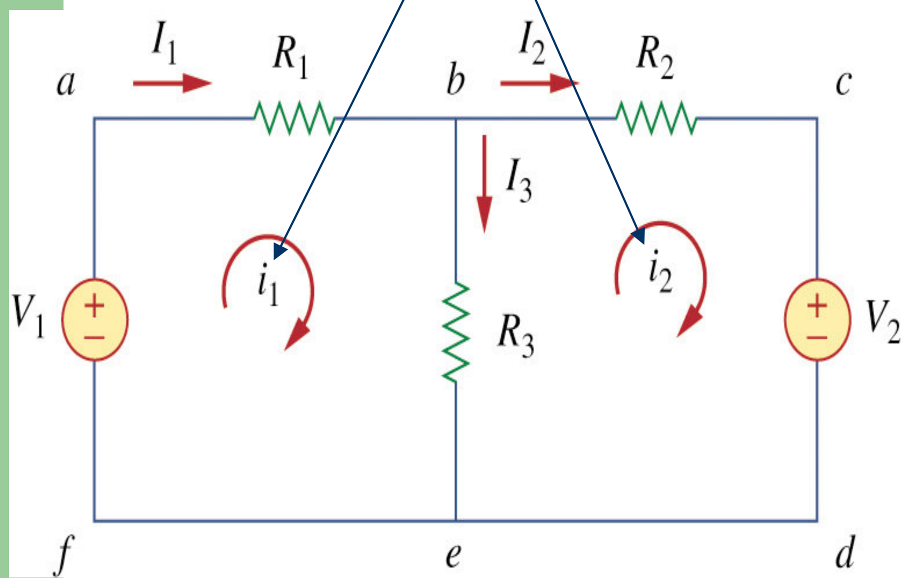
Mesh Current Example

Step 2: Apply KVL

Step 1: Assign Mesh Currents

Mesh 1: $-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0$

Mesh 2: $R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0$



Step 3: Solve the mesh

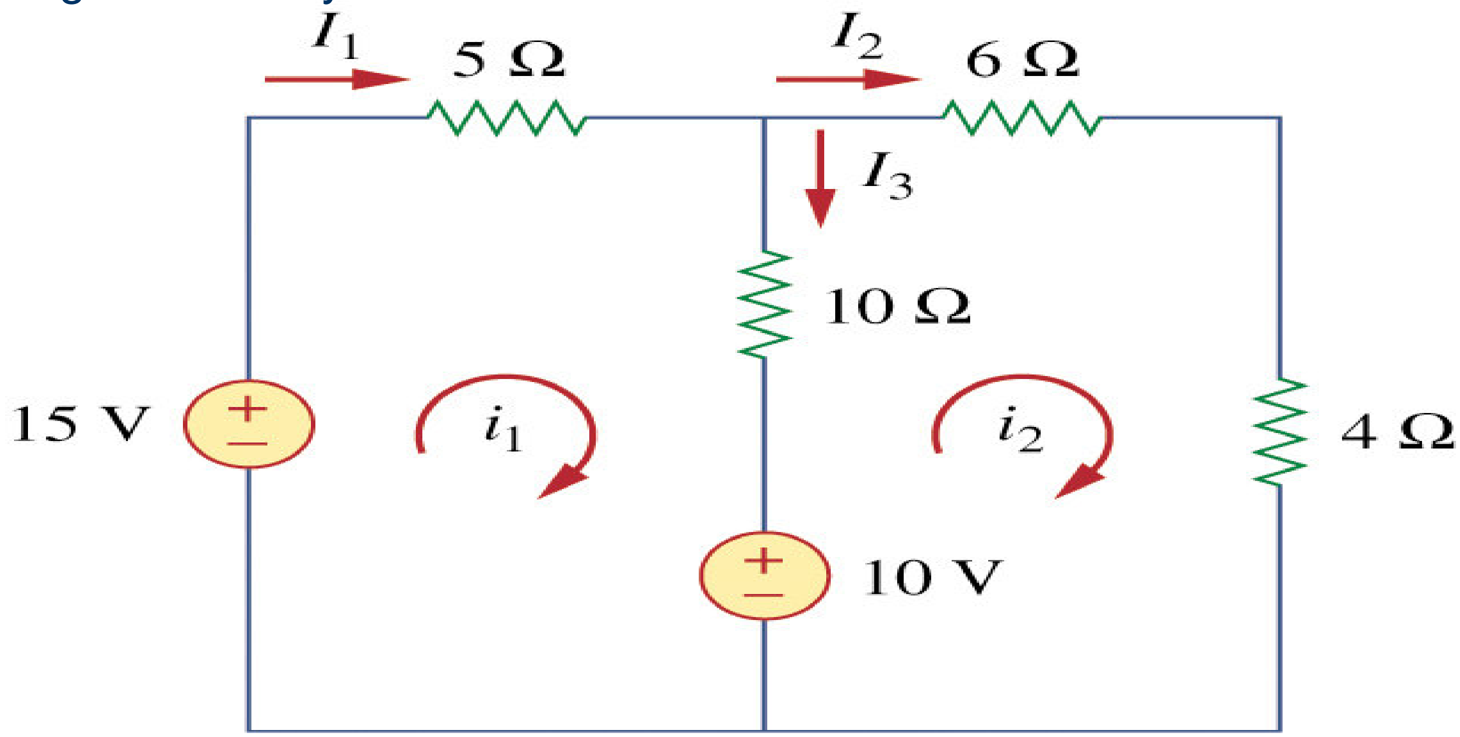
$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

Branch Current (I) $\neq$ Mesh Current (i)

$I_1 = i_1$, $I_2 = i_2$, $I_3 = i_1 - i_2$

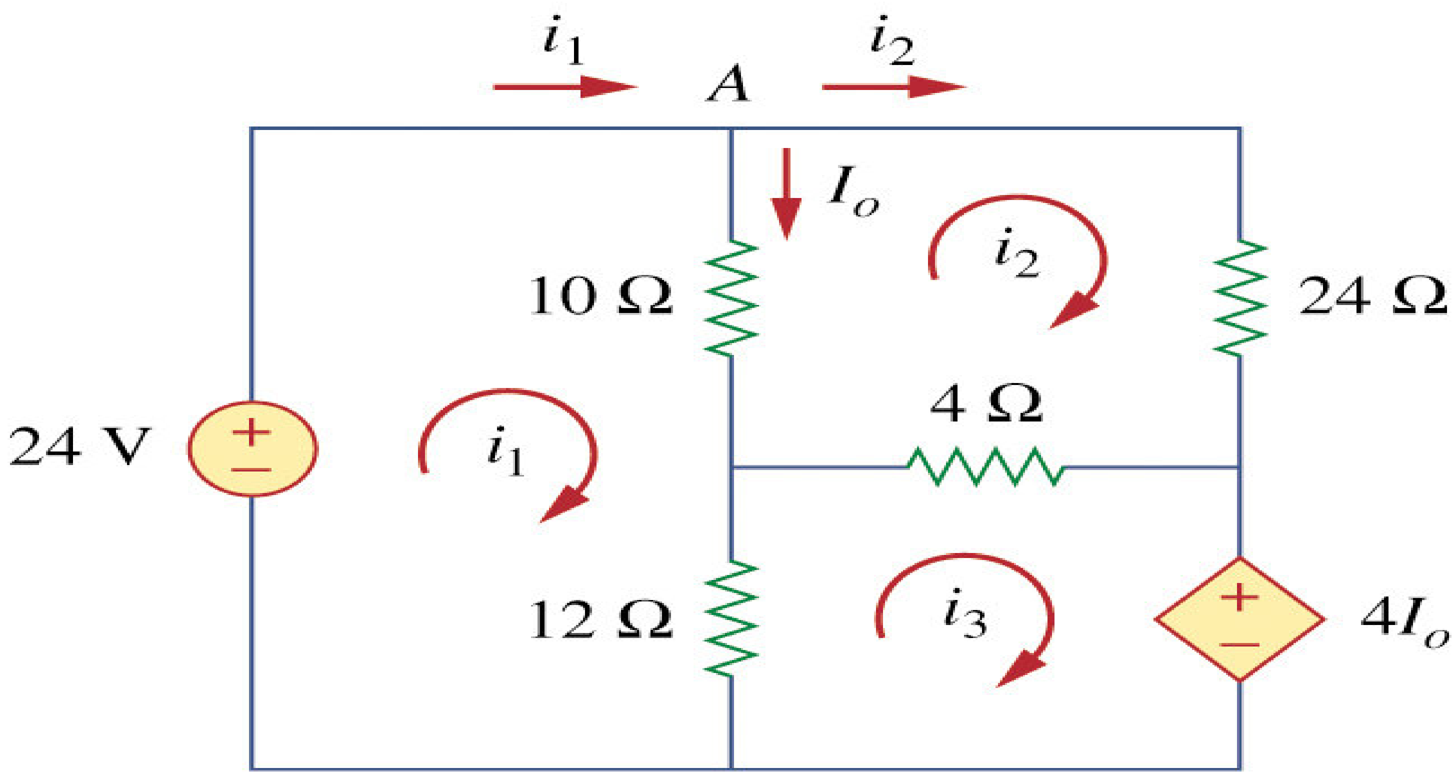
Example

For the circuit in below figure, find the branch currents I_1, I_2 and I_3 using mesh analysis.



Example

Use mesh analysis to find the current I0 in the below figure

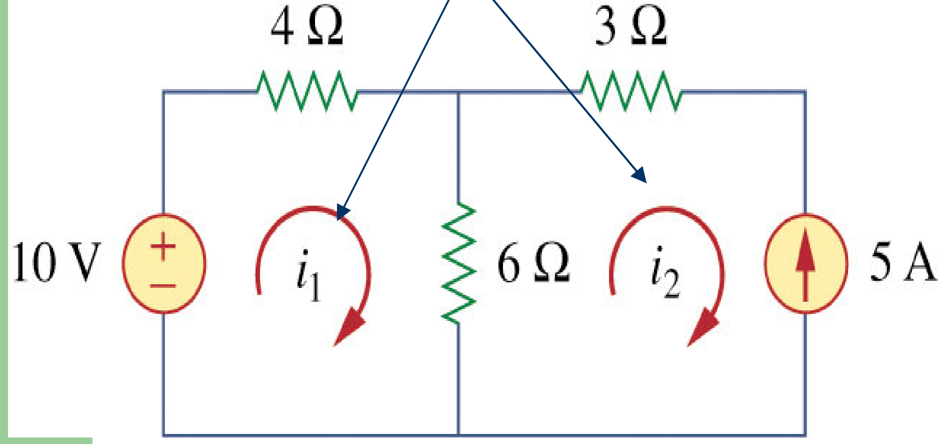


Determine Mesh Currents With Current Source (Mesh Analysis)

Situation 1

When a current source exists only in one mesh.

Step 1: Mesh Currents



Step 2: Apply KVL

$$-10 + 4i_1 + 6(i_1 - i_2) = 0$$

Set i_2 = -5A

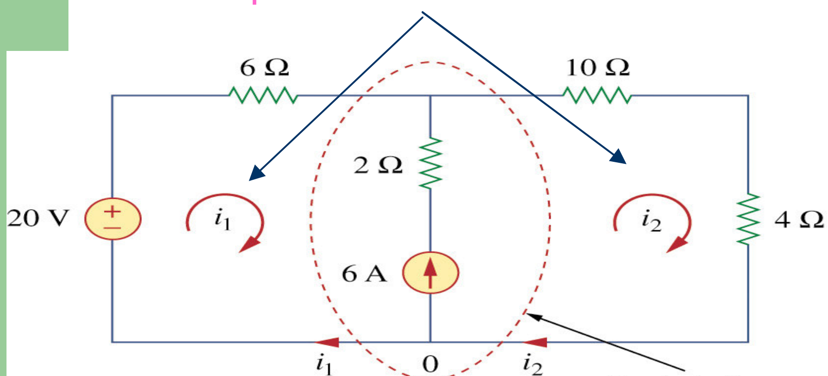
i_1 = -2A

Determine Mesh Currents With Current Source (Mesh Analysis)

Situation 2

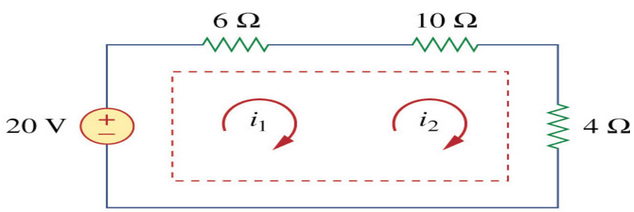
When a current source exists between two meshes. Create a supermesh by excluding the current source and any elements connected in series with it

Step 1: Mesh Currents



(a)

Step 2: Supermesh



(b)

Step 3: Apply KVL

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$

Step 4: Apply KCL to a node in the branch where 2 meshes intersect

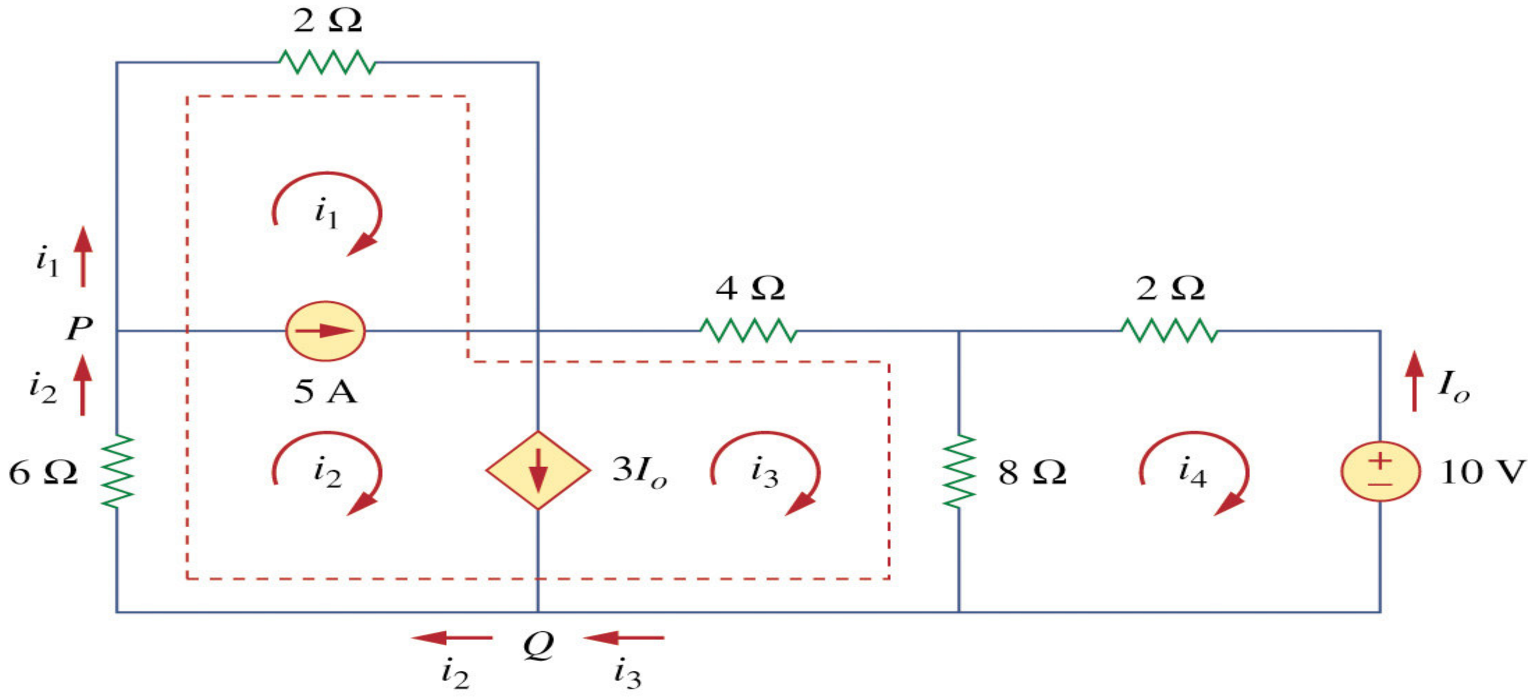
$$i_2 = i_1 + 6$$

Step 5: Solve it

$$i_2 = 2.8\text{A} \qquad i_1 = -3.2\text{A}$$

Example

Find i_1 and i_4 using mesh analysis



Summary

Nodal Analysis with Voltage Sources (Properties of a supernode)

- The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages
- A supernode has no voltage of its own
- A supernode requires the applicatiin of both KCL and KVL

Mesh Analysis with Current Sources (Properties of a supermesh)

- The current source in the supermesh provides the constraint equation necessary to solve for the mesh currents
- A supermesh has no current of its own
- A supermesh requires the application of both KVL and KCL

Nodal or Mesh Analysis?

Either choose method to reduce no. of equations:

- Many series connected elements → mesh
- Parallel-connected → nodal
- Fewer nodes than meshes → nodal

Or choose method to give answer easiest:

- Looking for node voltages → nodal
- Looking for branch/mesh current → mesh

IV-Circuit Theorems

Linearity Property

Linearity is the property of an element describing a linear relationship between cause and effect.

This property is a combination of both the *homogeneity* and *additivity* property

Homogeneity

If the input is multiplied by a constant, then the output is multiplied by the same constant

$$kiR = kv$$

Additivity

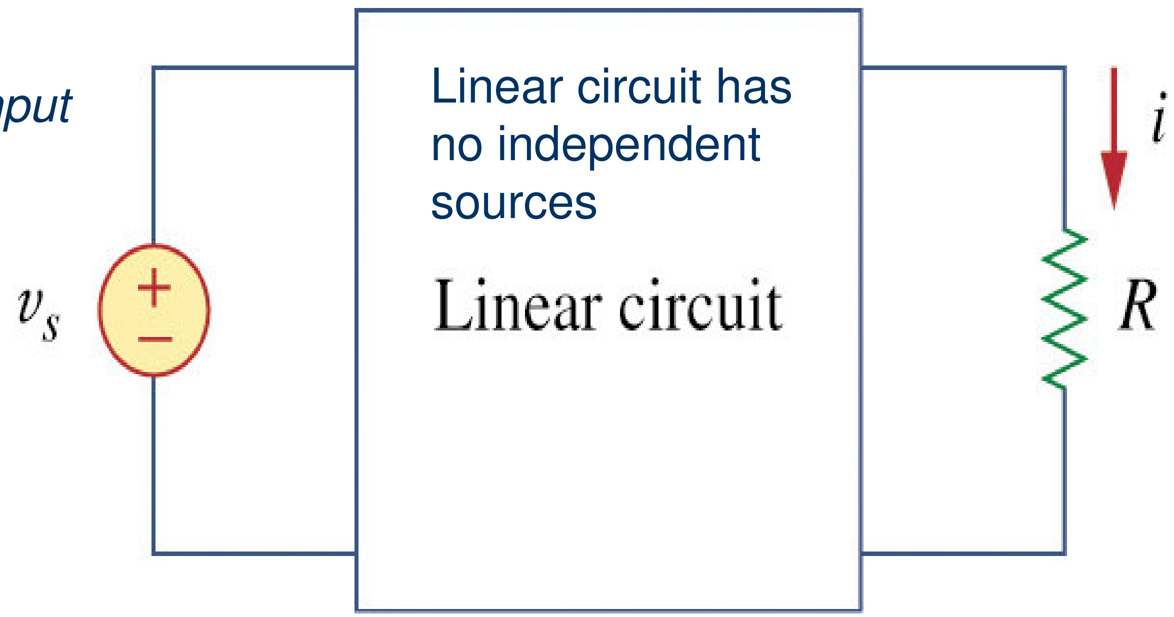
The response to a sum of input is the sum of the responses to each input applied separately

$$\begin{aligned} v &= (i_1 + i_2)R \\ &= i_1 R + i_2 R \\ &= v_1 + v_2 \end{aligned}$$

Note: For this chapter, the linearity property is limited to the resistor.

Example of a Linear Circuit

V_s as input voltage source



current *i* through R as the output

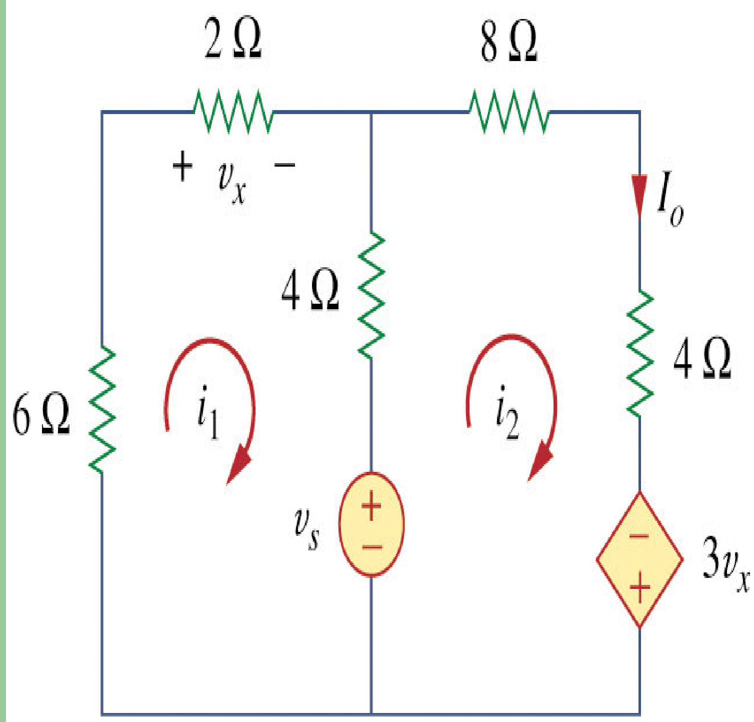
If v_s = 10V gives i = 2A, then according to linearity principle ,

when v_s= 1V, will give i=0.2A

(Reminder: Describing a linear relationship between cause and effect)

Example

For the circuit below, find I_o when v_s=12V and v_s=24V



Applying KVL to the 2 loops,

$12i_1 - 4i_2 + v_s = 0$ ----------------------(1)

$-4i_1 + 16i_2 - 3v_x - v_s = 0$ ----------------------(2)

given $v_x = 2i_1$, eq (2) becomes

$-10i_1 + 16i_2 - v_s = 0$ ----------------------(3)

Adding eq (1) and eq (3)

$2i_1 + 12i_2 = 0$---------------------$i_1 = -6i_2$ ------------(4)

Substituting eq (4) into eq (1)

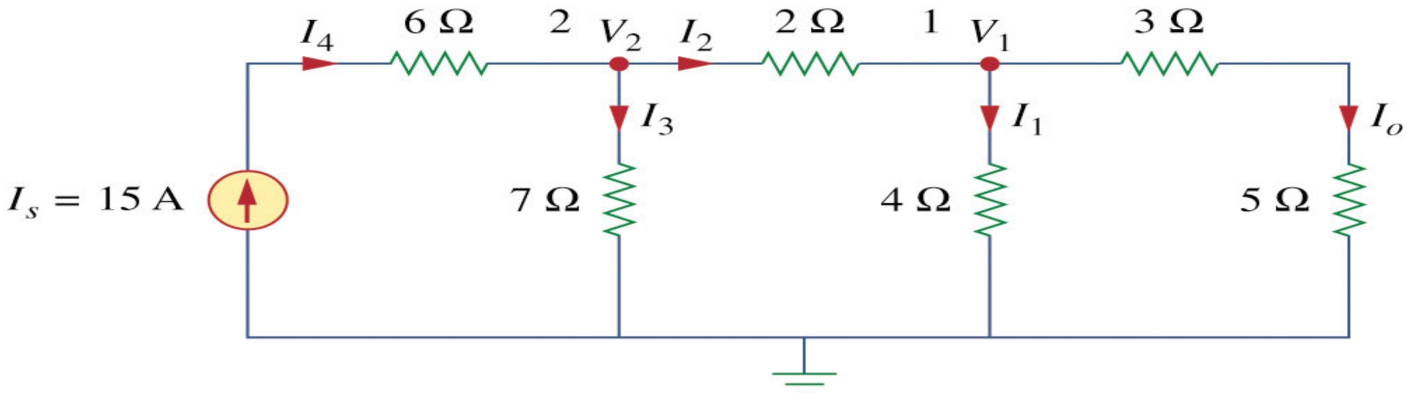
$-76i_2 + v_s = 0$---------------------$i_2 = v_s/76$

When v_s=12V, $I_o = i_2$=12/76A

When v_s=24V, $I_o = i_2$=24/76A (I_o doubles)

Example

Assume I_o = 1A and use linearity to find the actual value of I_o in the circuit below



node 1

Given I_o = 1A,

$V_1=(3+5)I_o=8(1)=8V$

$I_1=V_1/4=8/4=2A$

KCL

$I_2-I_o-I_1=0$, $I_2-1-2=0$, $I_2=3A$

node 2

Given $V_2-V_1=2I_2$

$V_2=2I_2-V_1$ $I_3=V_2/7=2A$

KCL

$I_4-I_3-I_2=0$, $I_4-2-3=0$ $I_4=5A=I_s$

If I_o= 1A, gives I_s= 5A

Then actual I_s= 15A, gives I_o= 3A

Superposition Principle

If a circuit has two or more independent sources, 1 way to determine the value of a specific variable (voltage or current is to use nodal or mesh analysis. Another way is to use superposition principle.

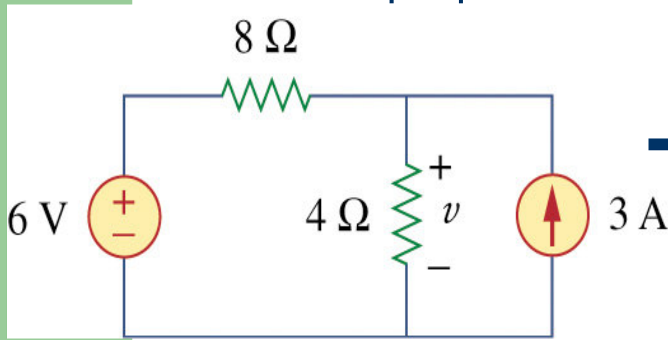
The superposition principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or current though) that element due to each independent source acting alone.

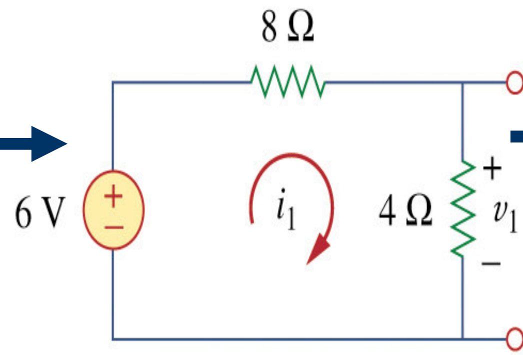
Steps to use superposition principle:

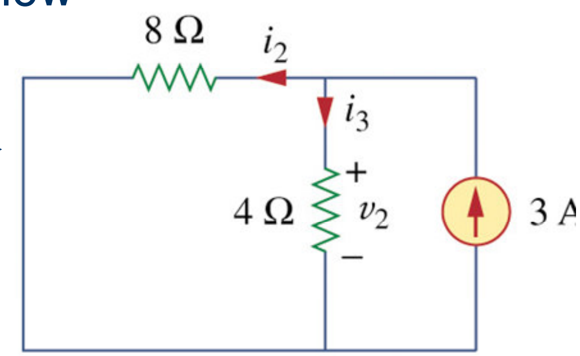
1. Turn off all independent sources except one source. Find the output (voltage or current) due to that active source using the techniques discussed before
2. Repeat step 1 for each of the other independent sources
3. Find the total contribution by adding algebraically all the contributions due to the independent sources

Example

Use the superpositon theorem to find *v* in the circuit below







There are 2 sources

Let $v = v_1 + v_2$

To obtain v_1, set the current source to zero

KVL

$12i_1 - 6 = 0$

$i_1 = 0.5A$

$V_1 = 4i_1 = 2V$

To obtain v_2, set the voltage source to zero

Using current division,

$i_3 = I\, R_2 / (R_2 + R_3)$

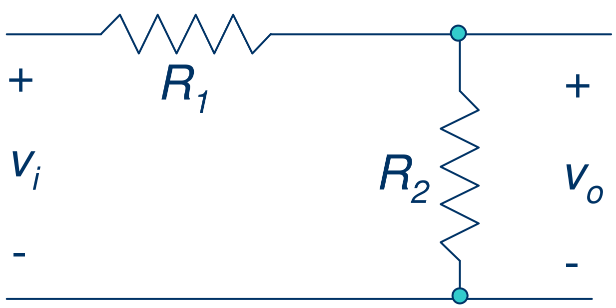
$= 3(8)/(8+4) = 2A$

$V_2 = 4i_3 = 4(2) = 8V$

Therefore $v = v_1 + v_2 = 2+8 = 10V$

Voltage or Current Divider

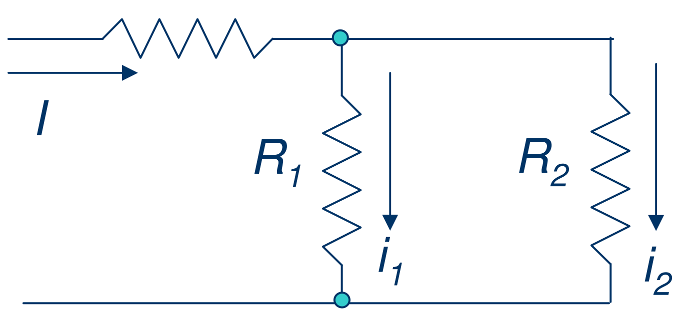
Voltage Divider



$$v_0 = v_i \frac{R_2}{R_1 + R_2}$$

$$\frac{v_o}{v_i} = \frac{R_2}{R_1 + R_2}$$

Current Divider

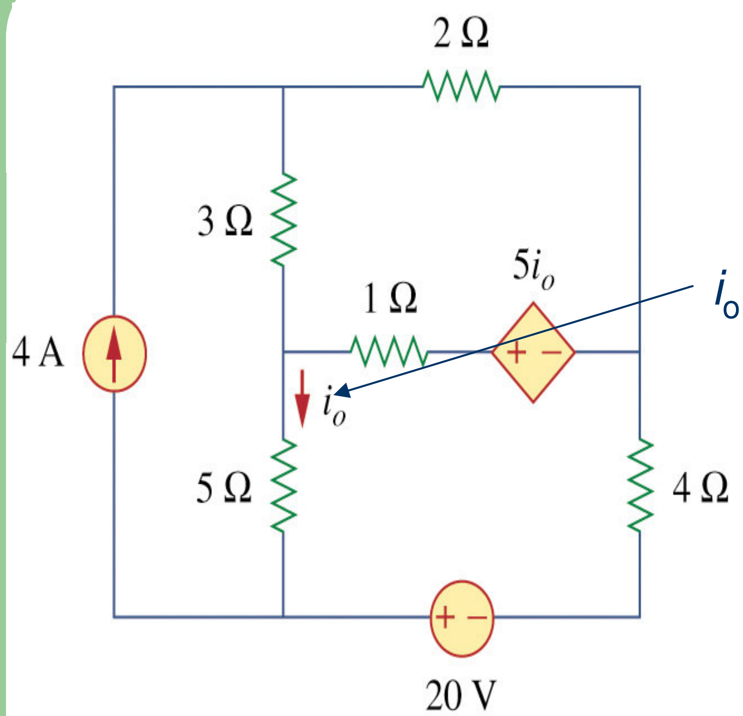


$$i_2 = I \frac{R_1}{R_1 + R_2}$$

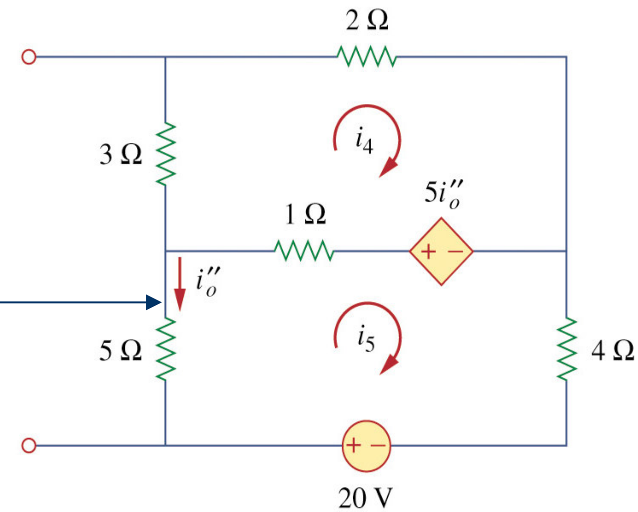
$$i_1 = I \frac{R_2}{R_1 + R_2}$$

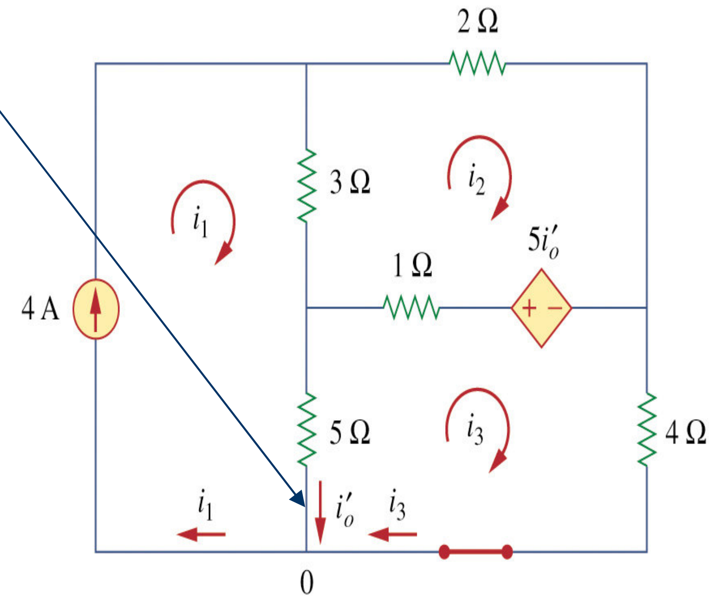
Example

Find i_o in the circuit below using superposition



$i_o = I'_o + I''_o$



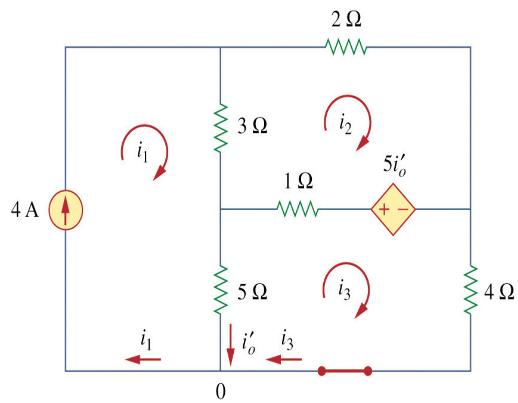


Steps to use superposition principle:

1. Turn off all independent sources except one source.

Cont…

To obtain i'_o turn off 20V



Loop 1: $i_1 = 4A$ -----(1)

Loop 2: $-3i_1+6i_2-1i_3-5i'_o=0$ -----(2)

Loop 3: $-5i_1-1i_2+10i_3+5i'_0$ -----(3)

At node 0: $i_3=i_1-i'_o= 4-i'_o$ -----(4)

Substitute eq (1) and eq (4) into (2)

$3i_2 – 2i'_o = 8$ -----(5)

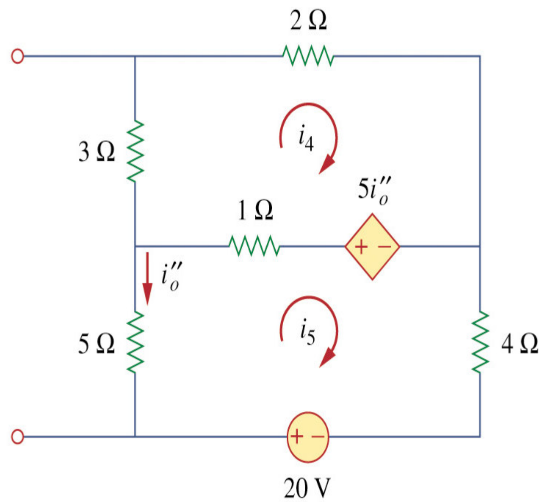
Substitute eq (1) and eq (4) into (2)

$I_2 + 5i'_o = 20$ -----(6)

Eliminate eq (5) and eq (6)

$i'_o = 52/17A$

To obtain I''_o turn off 20V



Loop 4: $6i_4-1i_5-5i''_o=0$ -----(7)

Loop 5: $-i_4+10i_5-20+5i''_0$ -----(8)

Given : $i_5 = -I''_o$ -----(9)

Substitute eq (9) and eq (7)

$6i_4– 4i''_o = 0$ -----(10)

Substitute eq (9) and eq (8)

$I_4 + 5i''_o = -20$ -----(11)

Eliminate eq (10) and eq (11)

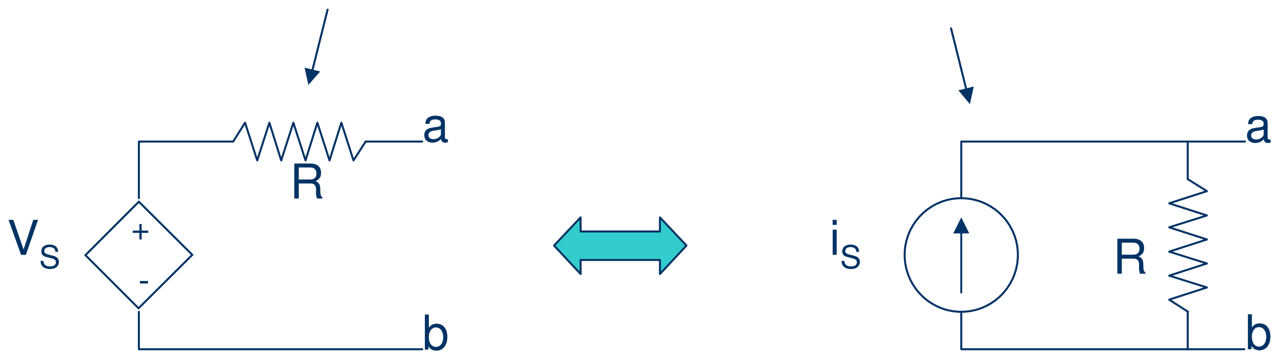
$I''_o = -60/17A$

Therefore: i_0= 52/17 -60/17 = -8/17A

Source Transformation

Source transformation is a tool for simplifying circuits. It is the process of replacing a voltage source v_s in series with a resistor R by a current source is in parallel with a resistor R_f or vice versa

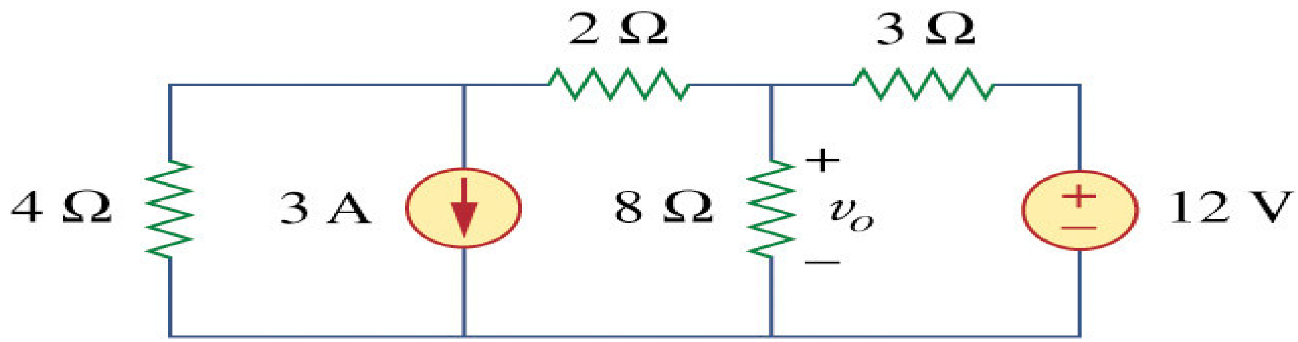
Change:
<u>*Resistor-Voltage in Series*</u> to <u>*Resistor-Current in Parallel*</u>

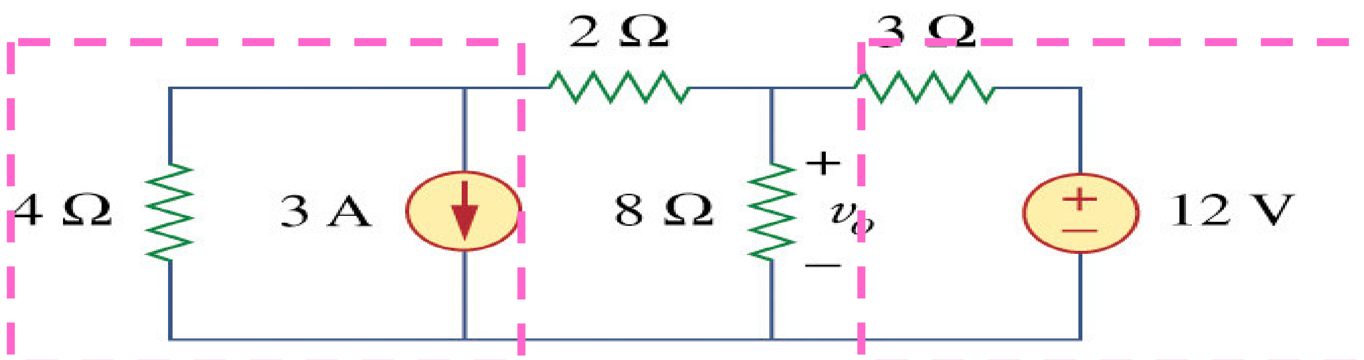


Note: Note possible where R=0 (ideal voltage source)

Example

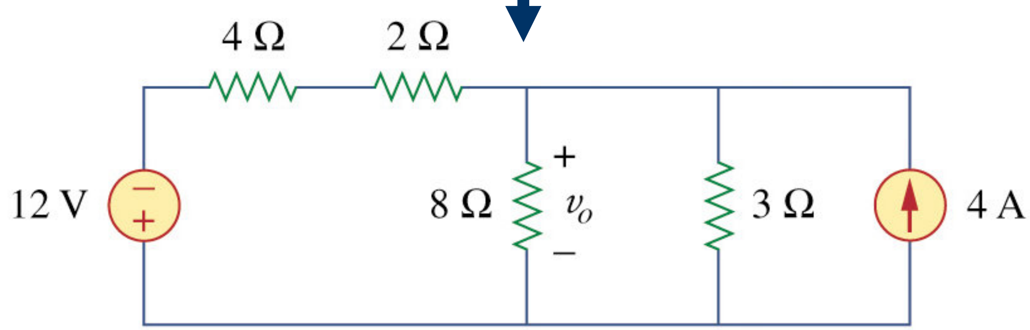
Use source transformation to find v_o in the circuit



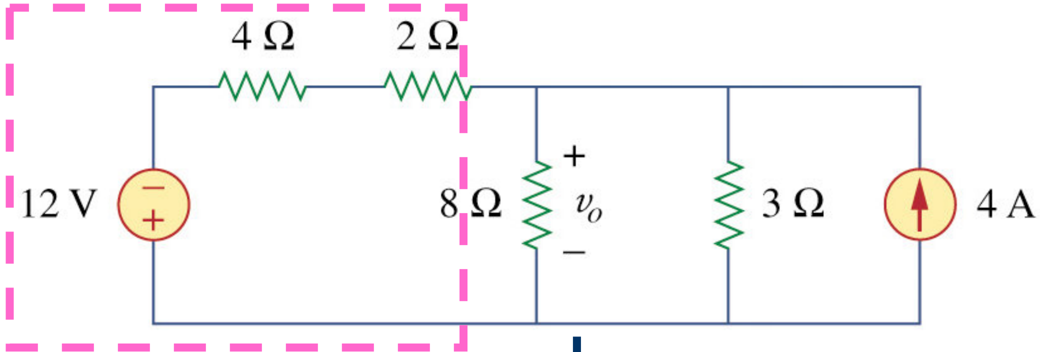


Replacing current source in parallel with resistor replace with a voltage source in series with resistor

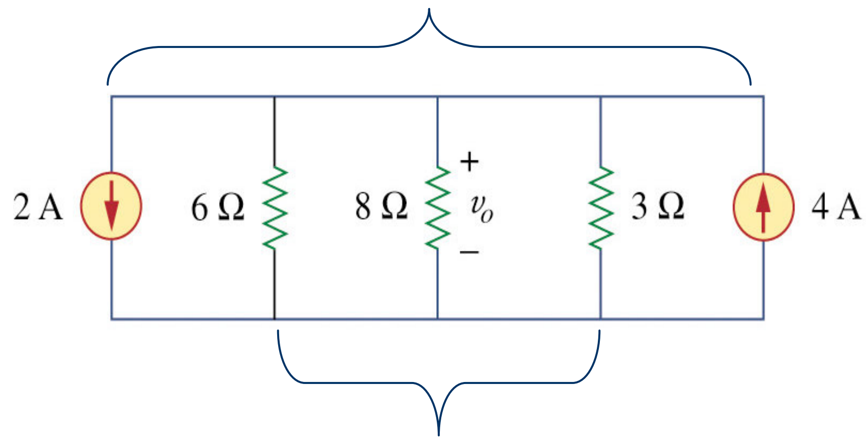
Replacing voltage source in series with resistor with current source in parallel with resistor



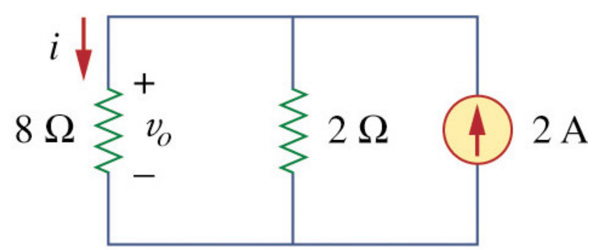
Cont…



Combine 2A and 4A



Combine 6 Ω and 3Ω in parallel



Current division

$$i = \frac{2}{2+8}(2) = 0.4\text{A}$$

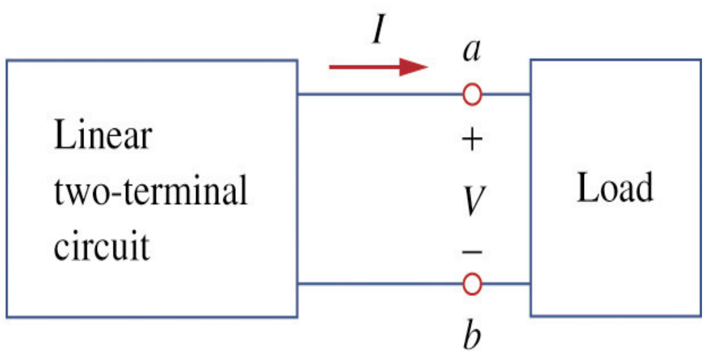
$$v_0 = 8i = 8(0.4) = 3.2\text{V}$$

Alternatively, since 8 Ω and 2Ω in parallel, they have the same voltage across them

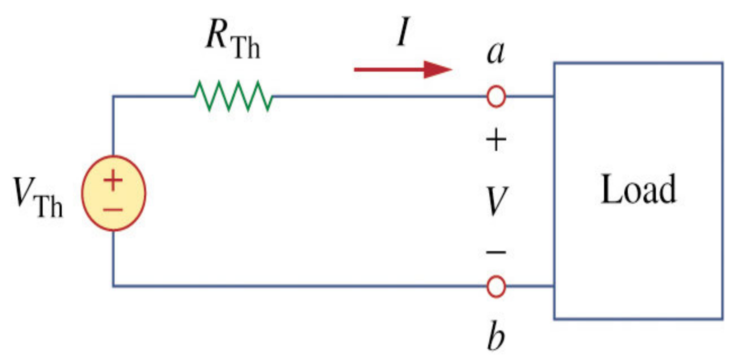
$$v_0 = (8\|2) = \frac{8\text{x}2}{10}(2) = 3.2\text{V}$$

Thevenin's Theorem

Thevenin's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{Th} in series with a resistor R_{Th}, where V_{Th} is the open-circuit voltage at the terminals and R_{Th} is the input or equivalent resistance at the terminals when the independent sources are turn off.



Original circuit

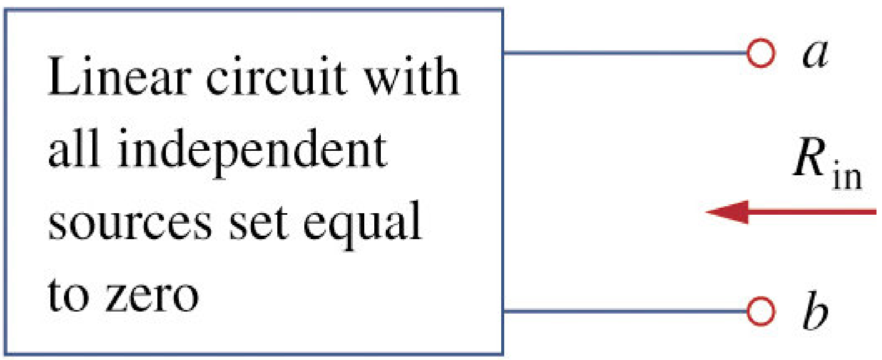


Thevenin equivalent circuit

2 Situations of Application of R_{Th}

Situation 1

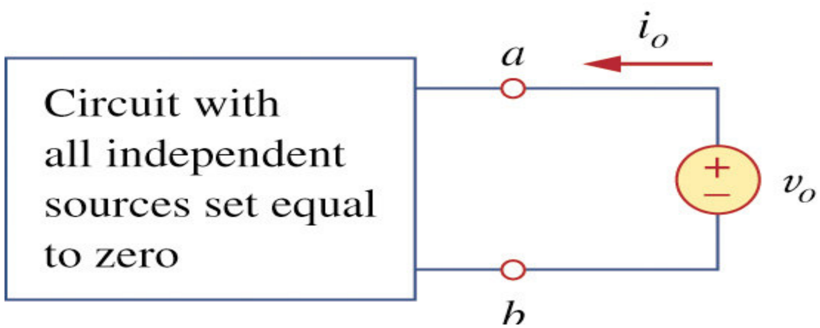
If network has no dependent sources, turn off all independent source. RTh is the input resistance of the network



$$R_{Th} = R_{in}$$

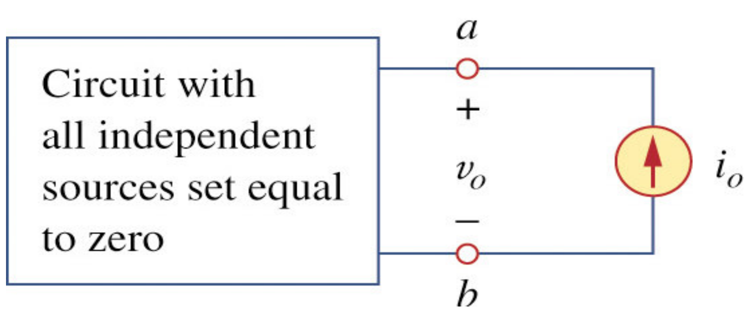
Situation 2

If network has dependent sources, turn off all independent source. Apply a voltage source v_0 at the terminals a and b and determine the resulting current i_o.



$$R_{Th} = \frac{v_o}{i_o}$$

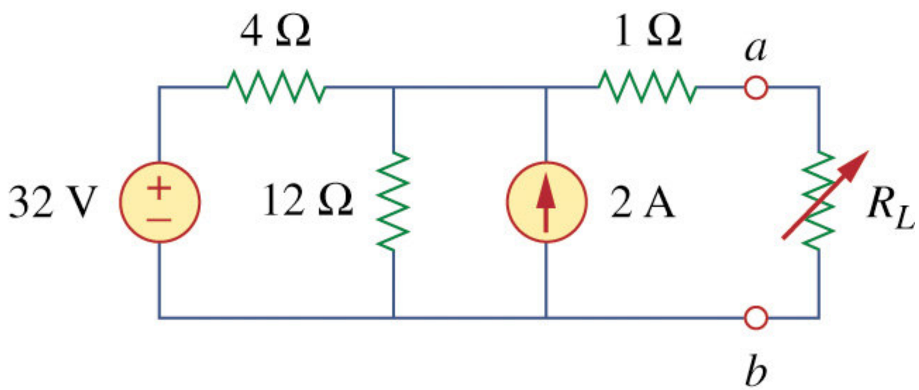
Alternatively, insert a current source i_o at terminals a-b and find the terminal voltage v_0.



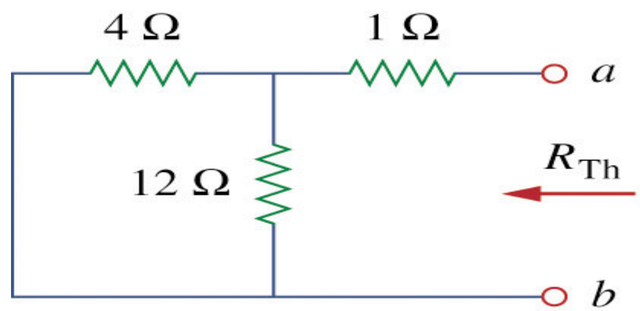
$$R_{Th} = \frac{v_o}{i_o}$$

Example

Find the Thevenin equivalent circuit shown below , to the left of the terminals a-b. Then find the current through R_L=6,16 and 36Ω.

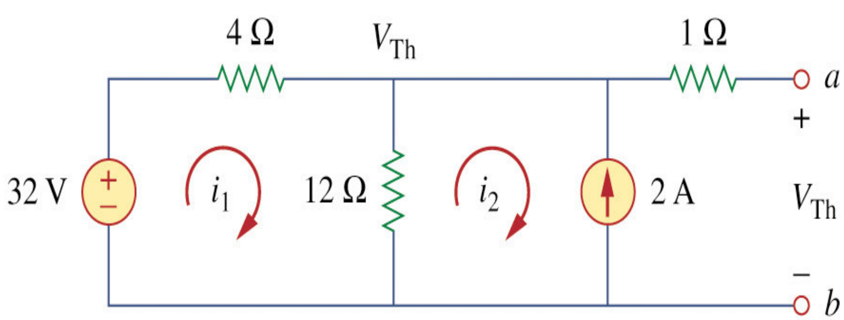


Step 1:Turn off all independent sources, R_{Th} is the input resistance of the network



$$R_{Th} = 4 \,\text{II}\, 12 + 1 = \frac{4X12}{16} + 1 = 4\Omega$$

Step 2:To find V_{Th}.Applying mesh analysis to the two loops.



$-32+4i_1+12(i_1-i_2)=0$ given i_2=-2A

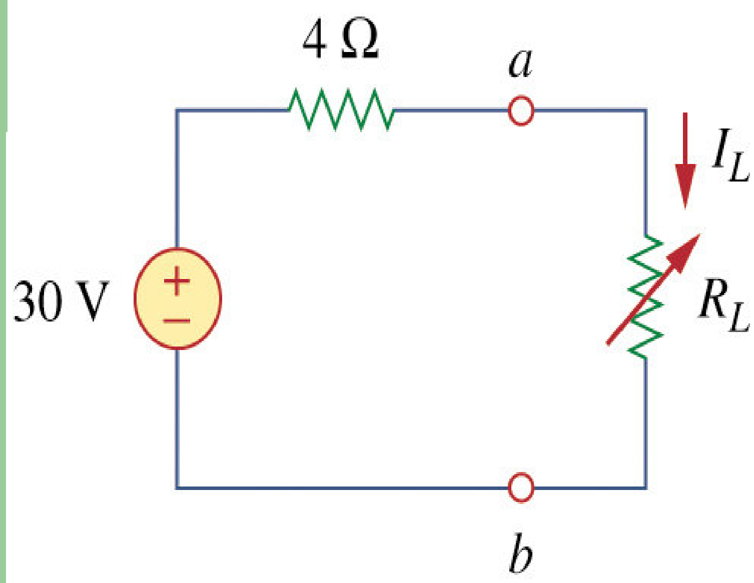
i_1=0.5A

VTh = $12(i_1-i_2)$=12(0.5+2)=30V

Cont…

Found $R_{Th} = 4\Omega$

$V_{Th} = 30V$



$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L}$$

$$when \quad R_L = 6$$

$$I_L = \frac{30}{10} = 3\text{ A}$$

$$when \quad R_L = 16$$

$$I_L = \frac{30}{20} = 1.5\text{ A}$$

$$when \quad R_L = 36$$

$$I_L = \frac{30}{40} = 0.75\text{ A}$$

Norton's Theorem

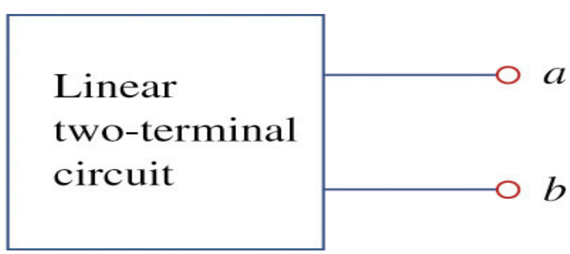
Norton's Theorem provides another equivalent circuit. It can be found by source-transformation of a Thevenin Circuit.

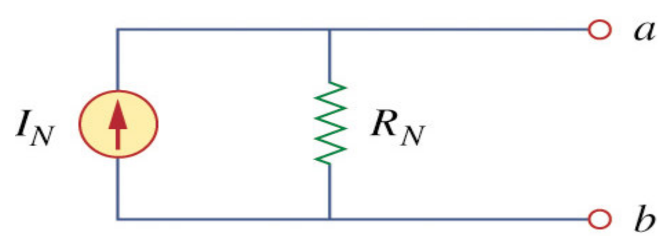
Linear two-terminal circuit →

Current Source I_N in Parallel with Resistor R_N

I_N – Short-circuit current through terminals

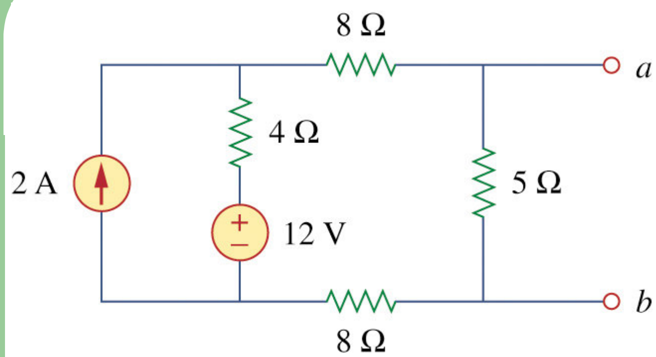
R_N – Equivalent resistance at terminal when <u>all the independent sources are turned off</u>





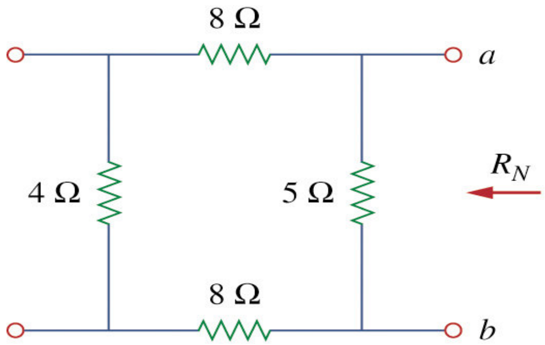
Example

Find the Norton Equivalent circuit in the circuit below at terminal a-b



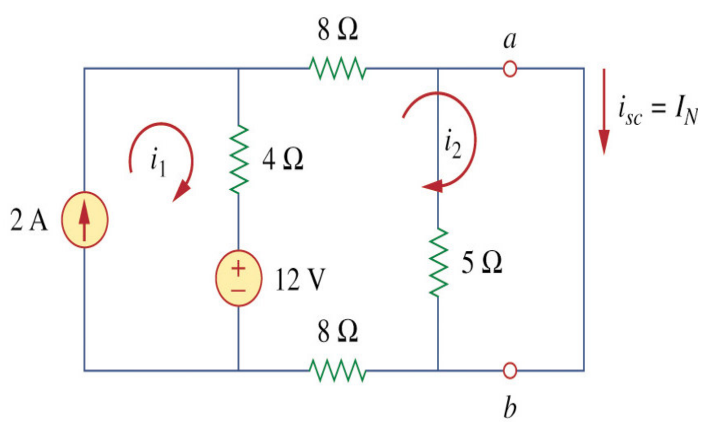
Find R_N in the same way as R_{Th}

Step 1: Set the independent source equal to zero



$$R_N = 5 \,\mathrm{II}\, (8+4+8) = 5 \,\mathrm{II}\, 20 = \frac{20\mathrm{X}5}{25} = 4\Omega$$

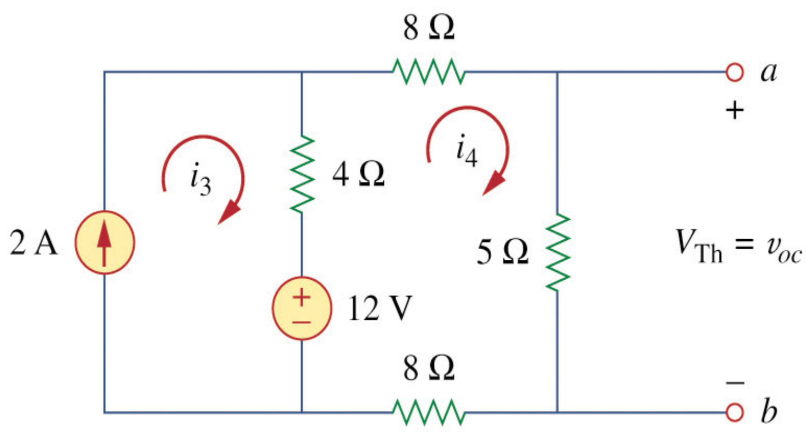
To find $I_{N,}$ short circuit terminals a and b.



$$i_1 = 2\mathrm{A}, \qquad 20i_2 - 12 - 4i_1 = 0$$

$$i_2 = 1\mathrm{A} = i_{\mathrm{sc}} = I_{\mathrm{N}}$$

Cont…



$$i_3 = 2\,\mathrm{A}$$

$$25\,i_4 - 4i_3 - 12 = 0 \qquad\qquad i_4 = 0.8\mathrm{A}$$

$$v_{oc} = V_{Th} = 5i_4 = 4\mathrm{V}$$

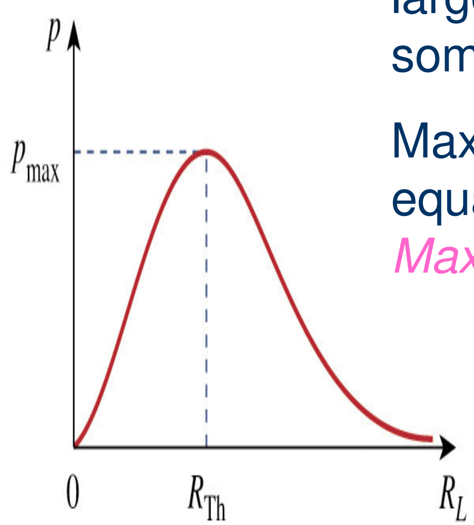
Hence

$$I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1\mathrm{A}$$

Maximum Power

Maximum power is transferred to the load when the load resistance equals the Thevenin resistance:

$R_L = R_{Th}$

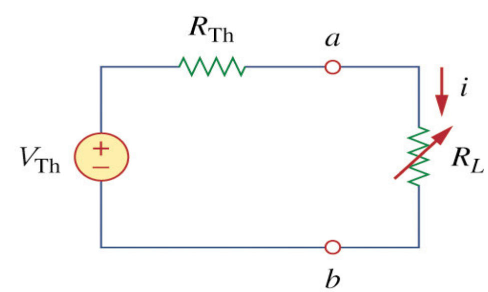


The power is small for a small or large value of R_L but maximum for some value of R_L between 0 and $\propto$

Maximum power occurs when RL is equal to R_{Th}. This is known as the *Maximum Power Theorem*

$$P_{\max} = \frac{V^2_{Th}}{4R_{Th}}$$

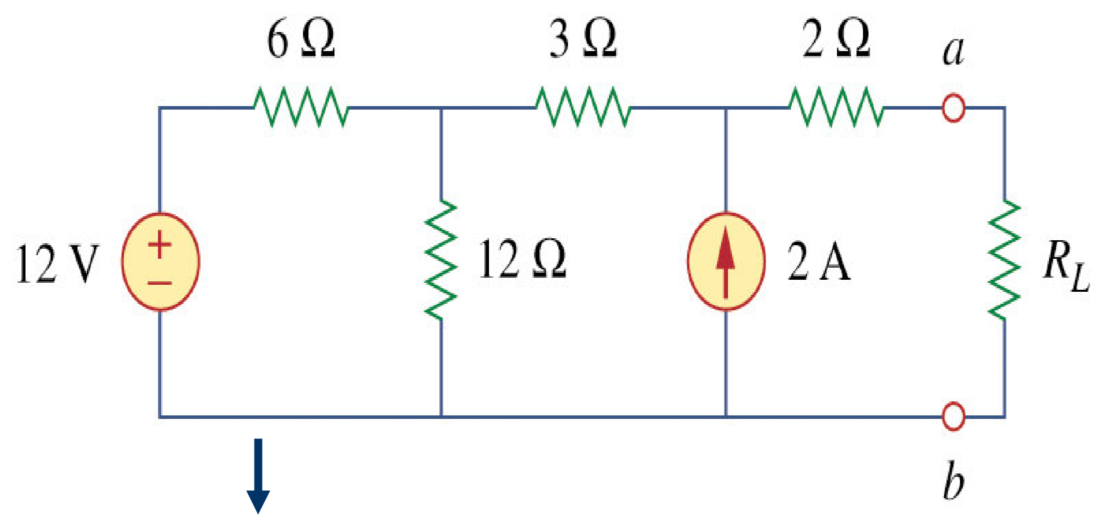
This is only applicable when $R_L = R_{Th}$



$$P = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L$$

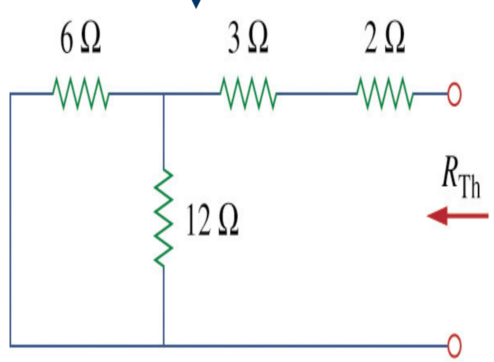
Example

Find the value of RL for maximum power transfer in the circuit below.

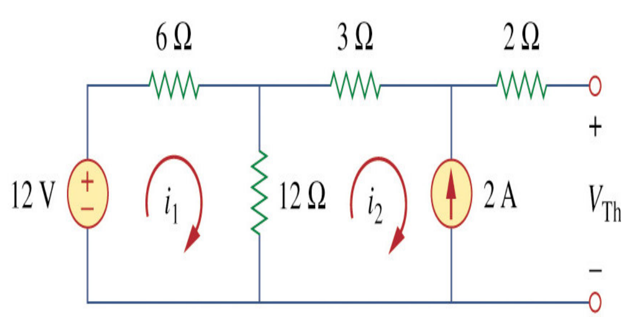


Step 1: Find R_{Th} and V_{Th}

Step 2: Set the independent source equal to zero



R_{Th} = 2+3+6 II 12= 5+ (6X12)/18 = 9Ω



$-12+18i_1-12i_2=0$ $\quad i_2=-2A$

$i_1=-2/3A$

$-12+6i_2+3i_2+2(0)+V_{Th}=0 \quad V_{Th}=22V$

$R_L = R_{Th}=9\Omega$

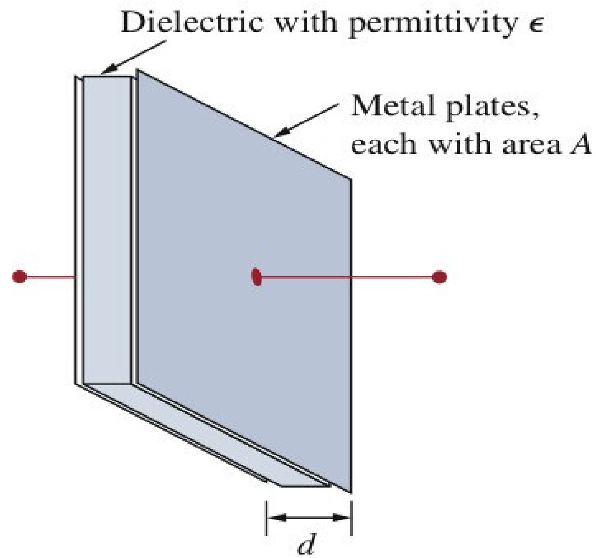
Maximum power is, $P_{max}= V^2Th/4R_L=22^2/4X9=13.44W$

V-Capacitors and Inductors

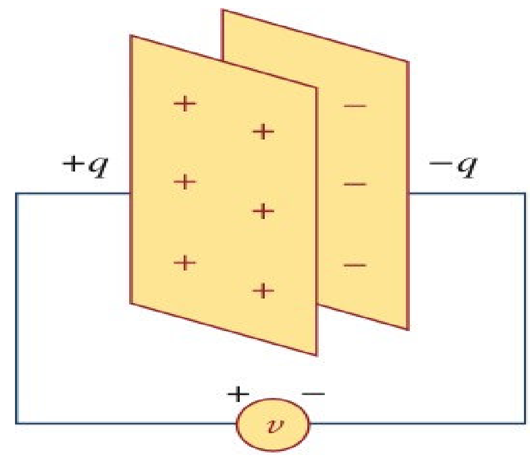
Capacitor

A passive element designed to store energy

A capacitor consists of two conducting plates separated by an insulator (or dielectric)



A typical capacitor



A capacitor with applied voltage v

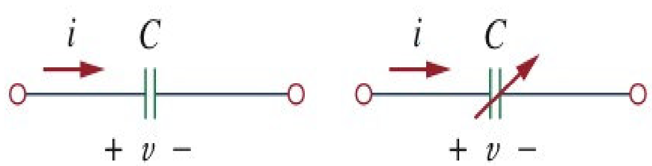
When a voltage source *v* is connected to the capacitor, the source deposits a +ve charge q on one plate and a –ve charge –q on the other.

$$q = Cv$$

q = amount charge store

C = Capacitance of the capacitor

v = applied voltage



Fixed capacitor Variable capacitor

Circuit symbols for capacitors

Capacitance

Capacitance is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F). However, capacitance depends on the physical dimensions of the capacitor, which is shown in following equation

$$C = \epsilon A/d$$

C = Capacitance

ϵ = Is the permittivity of the dielectric material between plate

A = Is the surface area of each plate

d = Is the distance between the plate

Note: According to the passive sign convention,

if $v>0$ and $i>0$ or if $v<0$ and $i<0$, the capacitor is being charged

if $v \cdot i < 0$, the capacitor is discharging

Voltage-Current Relationship for a Capacitor

Capacitor function:

Variable capacitors are used in radio receivers allowing one to tune to various stations. In addition, capacitors are used to block dc, pass ac, shift phase, store energy, start motors, and suppress noise

Current-voltage relationship of the capacitor:

$i = \mathrm{d}q/\mathrm{d}t$ $\qquad$ $q = Cv$

$i = C\mathrm{d}v/\mathrm{d}t$

Voltage-current relationship of the capacitor:

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,\mathrm{d}t \qquad \text{or} \qquad v = \frac{1}{C}\int_{-\infty}^{t} i\,\mathrm{d}t + v(t_0)$$

where $v(t_0) = q(t_0)/C$ is the voltage across the capacitor at the time t_0

The instantaneous power delivered to the capacitor is

$$p = vi = Cv\frac{\mathrm{d}v}{\mathrm{d}t}$$

The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p\,dt = C\int_{-\infty}^{t} v\frac{dv}{dt}dt = C\int_{-\infty}^{t} v\,dv = \frac{1}{2}Cv^2\Big|_{t=-\infty}^{t}$$

Note $v(-\infty) = 0$, because the capacitor was uncharged at $t = -\infty$

$$w = \frac{1}{2}Cv^2 \qquad \text{or} \qquad w = \frac{q^2}{2C}$$

both represent the energy stored in the electric field that exits between the plates of the capacitor

Capacitor's Properties

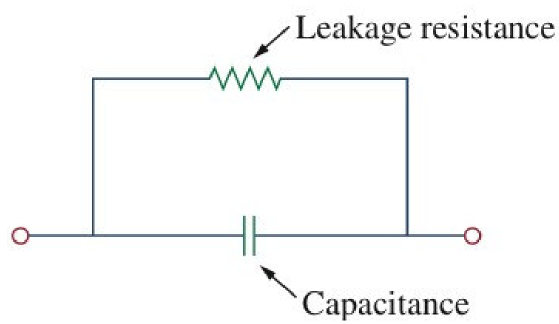
$$i = C\, dv/dt$$

Above equation shows that when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus, a capacitor is an open circuit to dc. If a battery (dc voltage) is connected across a capacitor, the capacitor charges.

The voltage on a capacitor must be continuous

Ideal capacitor does not dissipate energy

A real, nonideal capacitor has a parallel-model leakage resistance. The leakage resistance may be as high as 100MΩ and can be neglected for most practical applications.



Nonideal capacitor

Example

(a) Calculate the charge stored on a 3pF capacitor with 20V across it.
(b) Find the energy stored in the capacitor

$$q = Cv$$

$$q = 3x10^{-12}\ x20 = 60\,\text{pC}$$

The energy stored in the capacitor is

$$w = \frac{1}{2}Cv^{2} = \frac{1}{2}x3x10^{-12}\ x400$$

$$= 600\text{pJ}$$

Example

What is the voltage across a 3μF capacitor if the charge on one plate is 0.12mC?How much energy is stored?

$$q = Cv$$

$$0.12 \, x10^{-3} = 3 \, x10^{-6} \, xv$$

$$v = \frac{0.12 \, x10^{-3}}{3 \, x10^{-6}}$$

$$= 40 \text{ V}$$

The energy stored in capacitor is

$$w = \frac{1}{2} Cv^{2} = \frac{1}{2} x 3 \, x10^{-6} \, x1600$$

$$= 2400\text{x} \quad 10^{-6} \text{ J}$$

$$= 2.4\text{mJ}$$

Example

The voltage across a 5μF capacitor is

$v(t)$ = 10 cos 6000tV

Calculate the current through it

By definition, the current is

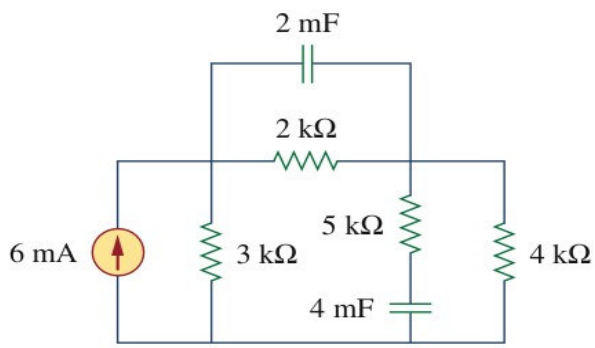
$$i(t) = C\frac{\mathrm{d}v}{\mathrm{d}t} = 5x10-6\frac{\mathrm{d}}{\mathrm{d}t}(10\cos 6000t)$$

$$= -5x10^{-6}x6000\,x10\sin 600t$$

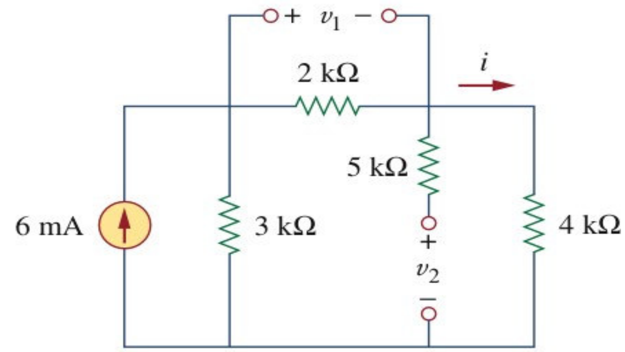
$$= -0.3\sin 6000t\,\mathrm{A}$$

Example

Obtain the energy stored in each capacitor in the circuit below under dc conditions



(a)



(b)

Under dc conditions, replace each capacitor with an open circuit, shown in figure (b). The current through the series combination of the 2kΩ and 4kΩ is obtained by current division as

$$i = \frac{3}{3+2+4}(6\text{mA})$$
$$= 2\text{mA}$$

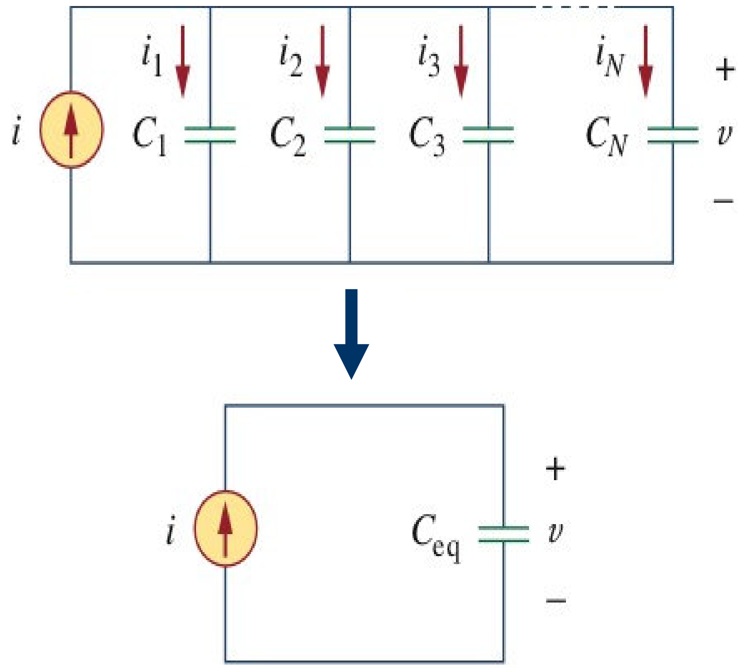
Hence, the voltage v_1 and v_2 across the capacitors are

$$v_1 = 2000\text{i} = 4\text{V}$$
$$v_2 = 400\text{i} = 8\text{V}$$

and the energies stored in them are

$$w_1 = \frac{1}{2}C_1 v_1^{\ 2} = \frac{1}{2}(2x10^{-3})(4)^2 = 16\,\text{mJ}$$
$$w_2 = \frac{1}{2}C_2 v_2^{\ 2} = \frac{1}{2}(4x10^{-3})(8)^2 = 128\,\text{mJ}$$

Parallel Capacitor



Note that the capacitor have the same voltage v across them.

Applying KCL,

$i = i_1 + i_2 + i_3 + \ldots + i_N$

But $i_k = C_k \, dv/dt$.

Hence

$$i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \ldots C_N \frac{dv}{dt}$$

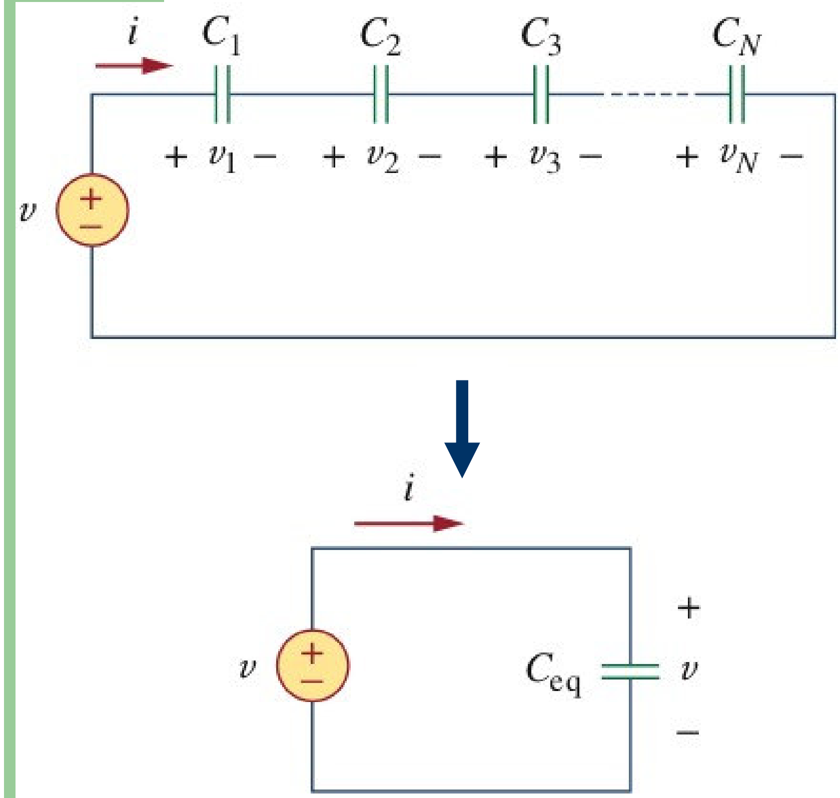
$$= \left(\sum_{k=1}^{N} C_k \right) \frac{dv}{dt} = Ceq \frac{dv}{dt}$$

where

$$C_{eq} = C_1 + C_2 + C_3 + \ldots + C_N$$

Note: Capacitor in parallel combine in the same manner as resistor in series

Series Capacitor



Applying KVL to the loop,

$$v = v_1 + v_2 + v_3 + ... + v_N$$

But $v_k = \frac{1}{C_k}\int_{t_o}^{t} i(t) + vk\,(t0)$. Therefore,

$$v_k = \frac{1}{C_1}\int_{t_0}^{t} i(t)\mathrm{d}t + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(t)\mathrm{d}t + v_2(t_0)$$

$$+ ... + \frac{1}{C_N}\int_{t_0}^{t} i(t)\mathrm{d}t + v_N(t_0)$$

$$= \left(\frac{1}{C_1} + \frac{1}{C_2} + ... + \frac{1}{C_N}\right)\int_{t_0}^{t} i(t)\mathrm{d}t + v_1(t_0) + v_2(t_0)$$

$$+ ... + V_N(t_o)$$

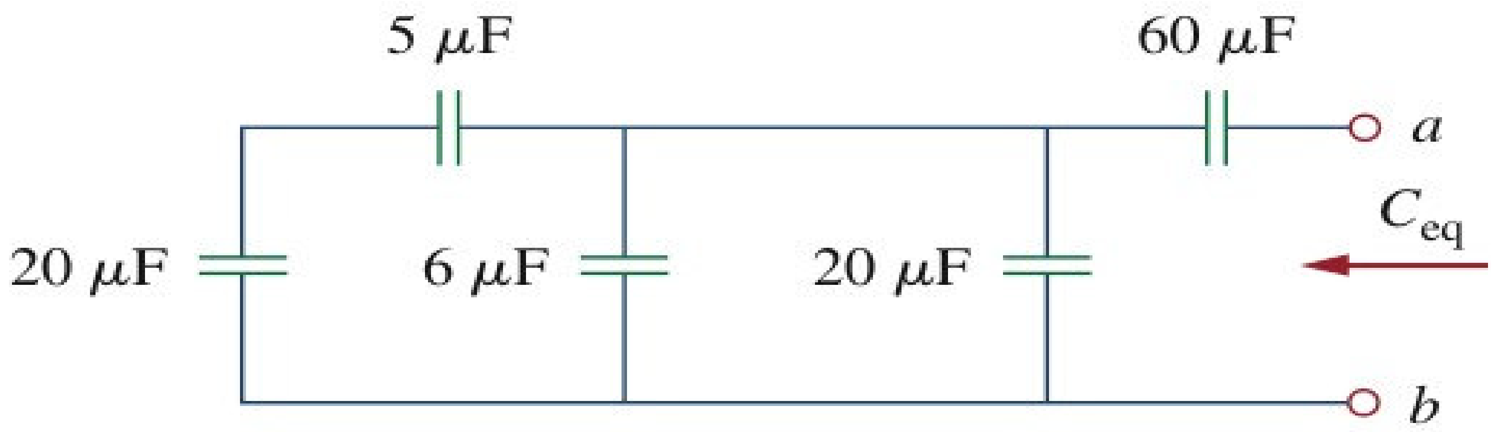
$$= \frac{1}{\mathrm{Ceq}}\int_{t_0}^{t} i(t)\mathrm{d}t + v(t_0)$$

where $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ... + \frac{1}{C_N}$

$$\mathrm{C}_{eq} = \frac{C_1 C_2 C_3 .... CN}{C_1 + C_2 + C_3 + ... C_N}$$

Example

Find the equivalent capacitance seen between terminal a and b of the circuit



The 20μF capacitor are in series; their equivalent capacitor is

$$\frac{20x5}{20+5} = 4\mu\text{F}$$

The 4μF is in parallel with the 6μF and 20μF capacitors; their combine capacitance is

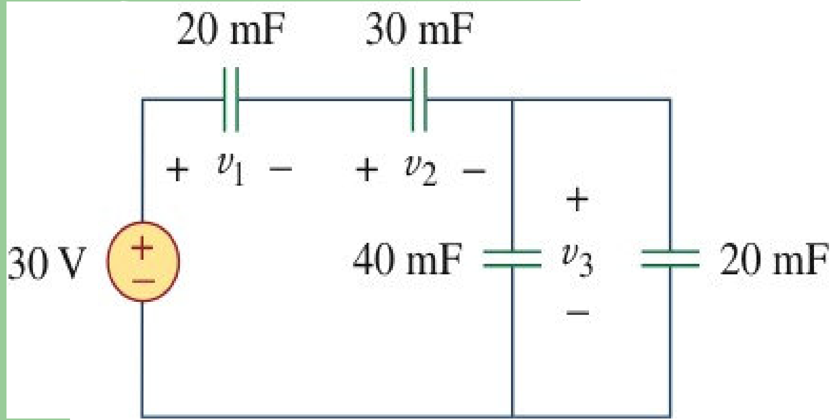
$$4+6+20 = 30\mu\text{F}$$

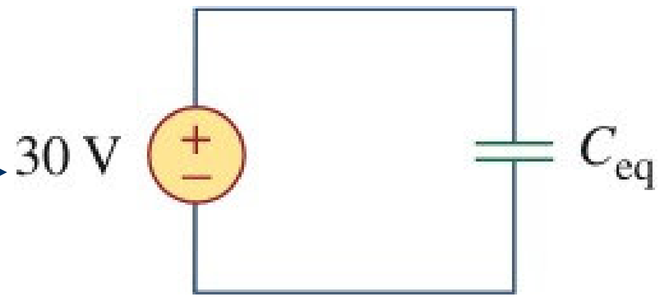
This 30μF is in series with the 60μF capacitor. Hence, the equivalent capacitance for the entire circuit is

$$C_{eq} = \frac{30x60}{30+60} = 20\mu\text{F}$$

Example

For the circuit below, find the voltage across each capacitor





The 2 parallel capacitors in Fig a can be combined to get 40+20=60mF. This 60mF capacitors is in series with the 20mF and 30mF capacitors. Thus

$$C_{eq} = \frac{1}{\frac{1}{60} + \frac{1}{30} + \frac{1}{20}} \text{mF} = 10 \text{ mF}$$

The total charge is

$$q = C_{eq} v = 10x10^{-3} x30 = 0.3\text{C}$$

Charge for v_1 and v_2

$$v_1 = \frac{q}{C_1} = \frac{0.3}{20\,x10^{-3}} = 15\text{ V}$$

$$v_2 = \frac{q}{C_2} = \frac{0.3}{30\,x10^{-3}} = 10\text{ V}$$

$$\text{KVL, } v_3 = 30 - v_1 - v_2 = 5\text{V}$$

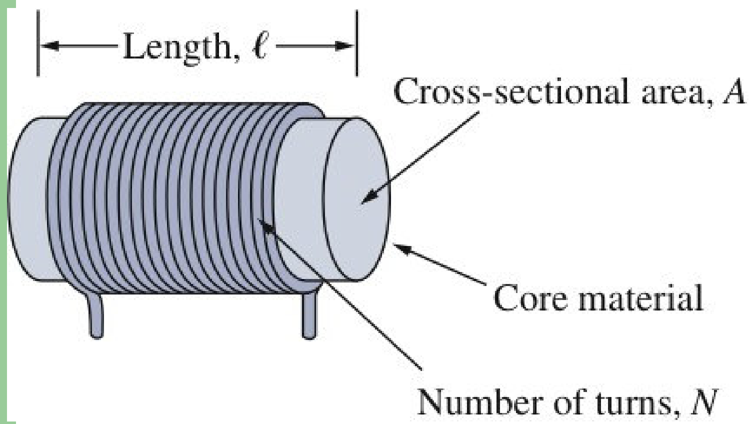
40mF and 20mF are parallel, have the same voltage as v_3 and the combined capacitance is 40+20=60mF

$$v_3 = \frac{q}{60\text{mF}} = \frac{0.3}{60x10^{-3}} = 5\text{V}$$

Inductor

A passive element designed to store energy in its magnetic field

An inductor consists of coil of conducting wire

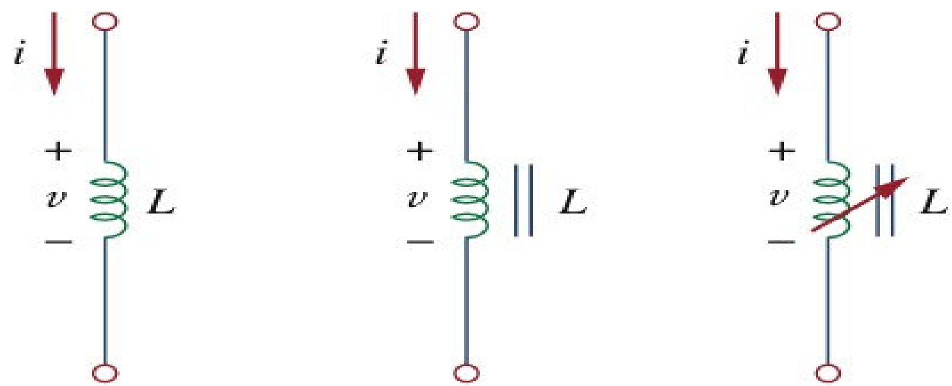


A typical form of an inductor

If current is allowed to through an inductor, if is found that the voltage across the inductor is directly proportional to the time rate of change of the current.

$$v = L\, di/dt$$

L = Inductance of the inductor



air-core

Iron-core

variable iron-core

Circuit symbols for inductors

Inductance

Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H)

$$L = \frac{N^2 \mu A}{\ell}$$

N = Number of turns
l = Is the length
A = Is the cross-sectional area
μ= Is the permeability of the core

Voltage-Current Relationship for an Inductor

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

$$\mathrm{d}i = \frac{1}{L}v\mathrm{d}t$$

$$i = \frac{1}{L}\int_{-\infty}^{t} v(t)\mathrm{d}t \qquad \text{or} \qquad i = \frac{1}{L}\int_{t_0}^{t} v(t)\mathrm{d}t + i(t_0)$$

Power delivered to the inductor is

$$p = vi = (L\frac{\mathrm{d}i}{\mathrm{d}t})i$$

The energy stored is

$$w = \int_{-\infty}^{t} pdt = \int_{-\infty}^{t} (L\frac{\mathrm{d}i}{\mathrm{d}t})idt = L\int_{-\infty}^{t} idi = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty)$$

since $i(-\infty) = 0$

$$w = \frac{1}{2}Li^2$$

Inductor's Properties

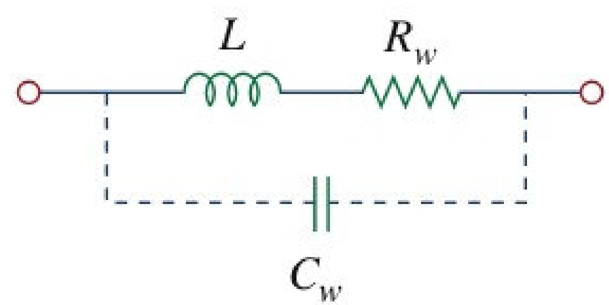
$v = L\, di/dt$

Show that when the voltage across an inductor is zero when the current is constant. Thus, an inductor acts like a short circuit to dc

The current through an inductor cannot change instantaneously

Ideal inductor does not dissipate energy

A real, nonideal has significant resistive component. This resistance is called the winding resistance R_w, and it appears in series with the inductance of the inductor. The presence of R_w makes it both an energy storage device and an energy dissipation device. The nonideal inductor also has a winding capacitance C_w due to the capacitive coupling between the conducting coils. C_w is very small and can be ignored.



Nonideal inductor

Example

The current through a 0.1H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

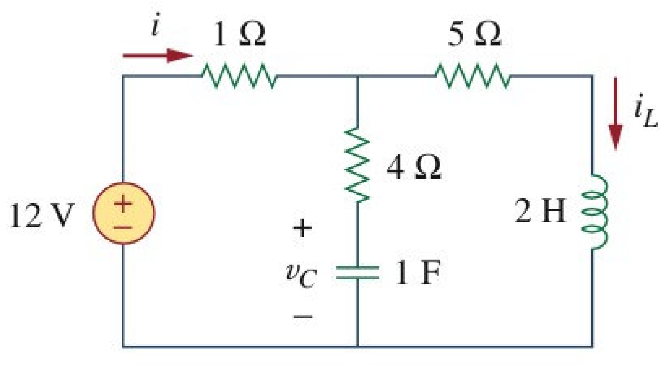
$$\text{Since } v = L\frac{\mathrm{d}i}{\mathrm{d}t} \text{ and } L = 0.1\mathrm{H}$$

$$v = 0.1\frac{\mathrm{d}}{\mathrm{d}t}\left(10\mathrm{te}^{-5t}\right) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1-5t)\mathrm{V}$$

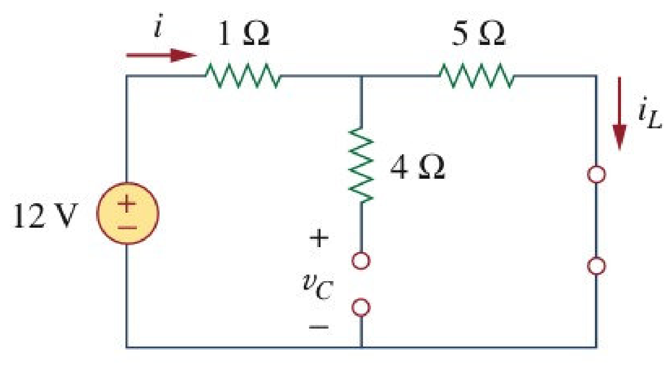
$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t}J$$

Example

Consider the circuit below. Under dc conditions, find: (a) i, v_c,and i_L (b) the energy stored in the capacitor and inductor



(a)



(b)

Under dc conditions, replace each capacitor with an open circuit, and the inductor with short circuit, as shown in figure (b).

$$i = iL = \frac{12}{1+5}$$
$$= 2\text{A}$$

The voltage vc is the same as the voltage across the 5Ω. Hence,

$$v_c = 5\text{i} = 10\text{V}$$

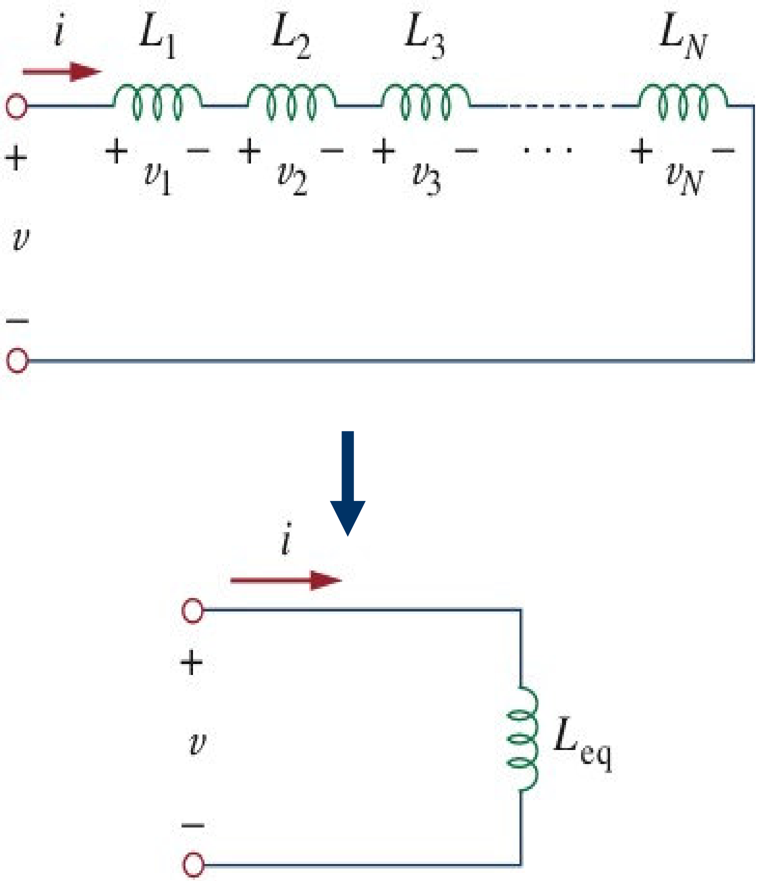
(b) The energy in the capacitor is

$$w_c = \frac{1}{2} C v_c{}^2 = \frac{1}{2}(1)(10)^2 = 50\,\text{J}$$

and that is the inductor is

$$w_L = \frac{1}{2} L i_L{}^2 = \frac{1}{2}(2)(2)^2 = 4\text{J}$$

Series Inductor



$$v = v_1 + v_2 + v_3 + \ldots + v_N$$

substituting $v_k = L_K \, \mathrm{d}i/\mathrm{d}t$

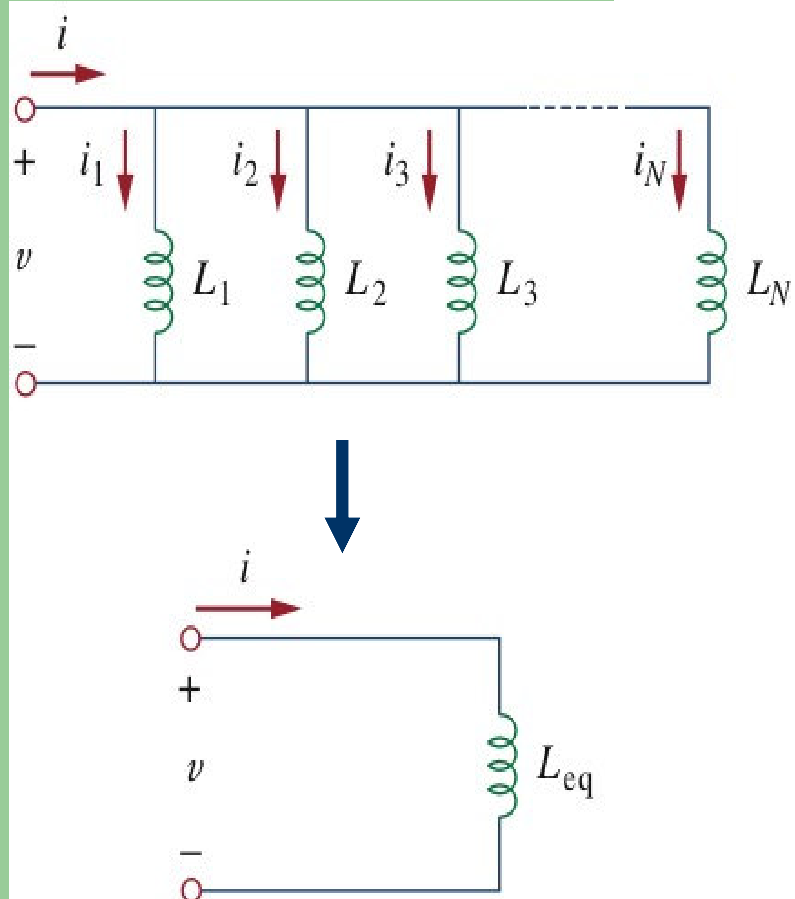
$$v = L_1 \frac{\mathrm{d}i}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i}{\mathrm{d}t} + L_3 \frac{\mathrm{d}i}{\mathrm{d}t} + \ldots + L_N \frac{\mathrm{d}i}{\mathrm{d}t}$$

$$= \left(L_1 + L_2 + L_3 + \ldots + L_N\right) \frac{\mathrm{d}i}{\mathrm{d}t}$$

$$= \left(\sum_{k=1}^{N} L_k\right) \frac{\mathrm{d}i}{\mathrm{d}t} = Leq \frac{\mathrm{d}i}{\mathrm{d}t}$$

where $L_{eq} = L_1 + L_2 + L_3 + \ldots + L_N$

Parallel Inductor



$$i = i_1 + i_2 + i_3 + \ldots + i_N$$

$$\text{But } i_k = \frac{1}{L_k}\int_{t_0}^{t} v\,dt + i_k\,(t_0)$$

$$i = \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1\,(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2\,(t_0)$$

$$+ \ldots + \frac{1}{L_N}\int_{t_0}^{t} v\,dt + i_N\,(t_0)$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\,dt + i_1\,(t_0) + \ldots + i_N(t_0)$$

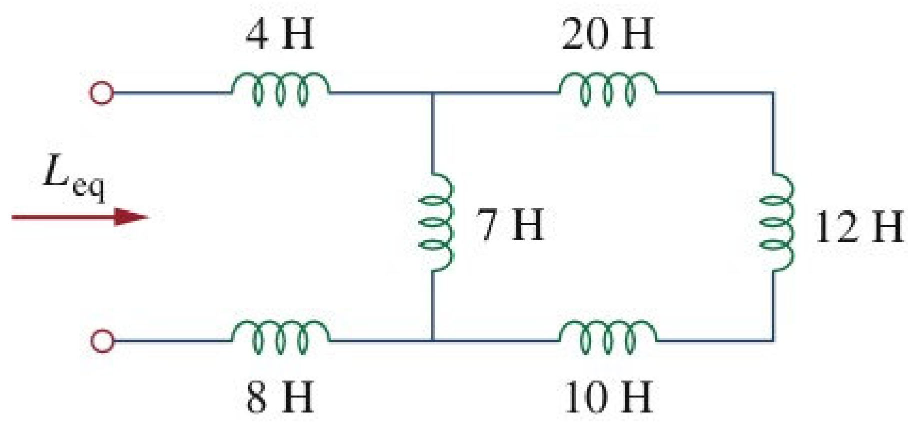
$$= \left(\sum_{k=1}^{N} \frac{1}{L_k}\right)\int_{t_0}^{t} v\,dt + \sum_{k=1}^{N} i_K(t_0)$$

$$= Leq\int_{t_0}^{t} v\,dt + i(t_0)$$

$$\text{where } \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \ldots + \frac{1}{L_N}$$

Example

Find the equivalent inductance of the circuit below



10H,12H and 20H are in series, thus combining them gives a 42H inductance. 42H inductor is in parallel with the 7H inductor so that they are combine, to give

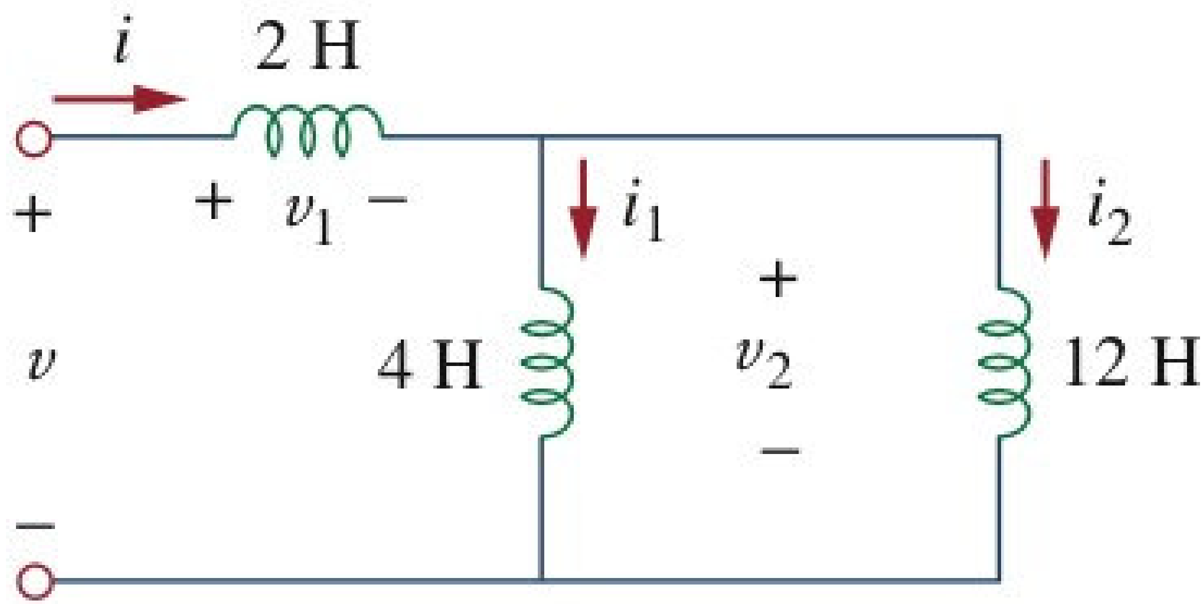
$$(7 \times 42)/(7+42)= 6H$$

The 6H inductor is in series with the 4H and 8H inductor. Hence,

$$L_{eq} = 4+6+8=18H$$

Example

For the circuit below, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find:
a) $i_1(0)$, b) $v(t)$, $v_1(t)$, and $v_2(t)$; c) $i_1(t)$ and $i_2(t)$



Cont…

From $i(t) = 4(2 - e^{-10t})\,\text{mA}$, $i(0) = 4(2 - 1) = 4\,\text{mA}$.

a) Since $i = i_1 + i_2$

$i_1(0) + i(0) - i_2(0) = 4 - (-1) = 5\ \text{mA}$

b) The equivalent inductance is

$L_{eq} = 2 + 4 \| 12 = 2 + 3 = 5\ \text{H}$

Thus,

$$v(t) = L_{eq}\frac{\mathrm{d}i}{\mathrm{d}t} = 5(-4)(-10)e^{-10t}\,\text{mV} = 200\,e^{-10t}\,\text{mV}$$

and

$$v_1(t) = 2\frac{\mathrm{d}i}{\mathrm{d}t} = 2(-4)(-10)e^{-10t}\ \text{mV} = 80\,e^{-10t}\,\text{mV}$$

since $v = v_1 + v_2$

$$v_2(t) = v(t) - v_1(t) = 120\,e^{-10t}\,\text{mV}$$

Cont…

The current i_1 is obtained as

$$i_1(t) = \frac{1}{4}\int_0^t v_2 \mathrm{d}t + i_2(0) = \frac{120}{4}\int_0^t e^{-10t}\mathrm{d}t + 5 \text{ mA}$$

$$= -3e^{-10t}\Big|_0^t + 5\text{mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t}\text{mA}$$

Similarly,

$$i_2(t) = \frac{1}{12}\int_0^t v_2 \mathrm{d}t + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\mathrm{d}t - 1 \text{ mA}$$

$$= -e^{-10t}\Big|_0^t - 1\text{mA} = -e^{-10t} + 1 - 1 = -e^{-10t}\text{mA}$$

Note that $i_1(t) = i_2(t) = i(t)$

Application

Capacitors and inductors possess the following three special properties that make them very useful in electric circuits:

- The capacity to store energy makes them useful as temporary voltage or current sources. Thus, they can be used for generating a large amount of current of voltage for a short period of time. (DC)
- Capacitors oppose any abrupt change in voltage, while inductors oppose any abrupt change in current. This property make inductors useful for spark or arc suppression and for converting pulsating dc voltage into relatively smooth dc voltage. (DC)
- Capacitors and inductors are frequency sensitive. This property makes them useful for frequency discrimination. (AC)

VI-First-Order Circuits

Introduction

2 types of of simple circuits will be examined:

a. A circuit comprising a resistor and a capacitor (RC circuit)

b. A circuit comprising a resistor and an inductor (RL circuit)

The analysis of *RC* and *RL* circuits can be performed by applying Kirchhoff's laws. However, by applying Kirchhoff's laws produces differential equation. The differential equations resulting from analysing RC and RL circuits are of the first order. Hence, the circuits are collectively known as first-order circuits.

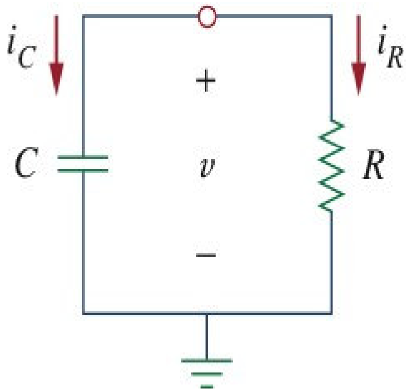
2 ways to excite the *RC* and *RL* circuits:

1st: Initial conditions – assume energy is in capacitative or inductive elements. Current flows in circuit and is dissipated. No independent sources allowed, dependent sources allowed.

2nd:Independent sources – for now we consider only DC sources which are excited

Source-Free *RC* Circuit

A source-free *RC* circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.



Initially the capacitor is charged, we can assume that at time t = 0, the initial voltage is

$$v(0) = V_0$$

$$i_c + i_R = 0$$

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

(1^{st} order differenti al eq)

Integrating both sides,

$$\ln v = -\frac{t}{RC} + \ln A$$

In A is the integration constant. Thus

$$\ln\frac{v}{A} = -\frac{t}{RC}$$

$$v(t) = Ae^{-t/RC}$$

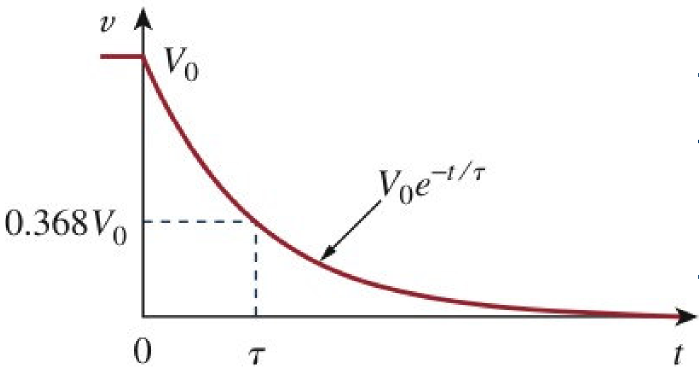
$$v(0) = A = V_0.$$

$$v(t) = V_0 e^{-t/RC}$$

This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. Since the response is due to initial energy stored and not external voltage, so it is called the natural response of the circuit

As t increases, the voltage decreases to zero and the rapidity with which the voltage decreases is expressed in term of the time constant (τ)

Cont…



The time constant (τ) of a circuit is the time required for the response to decay to a factor of 1/e or 36.8 percent of its value.

This implies that at t=τ

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368 V_0$$

or

$$\tau = RC$$

In term of time constant, voltage response of the RC circuit can be written as

$$v(t) = V_0 e^{-t/\tau}$$

$$i(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau}$$

The power dissipated in the resistor is

$$p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau}$$

The energy absorbed by the resistor up to *t* is

$$w_R(t) = \int_0^t p\,dt = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau} dt$$

$$= -\frac{\tau V_0^2}{2R} e^{-2t/\tau} \Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau})$$

Source-free RC circuit Summary

The Key to working with a source-free *RC* circuit is to find:

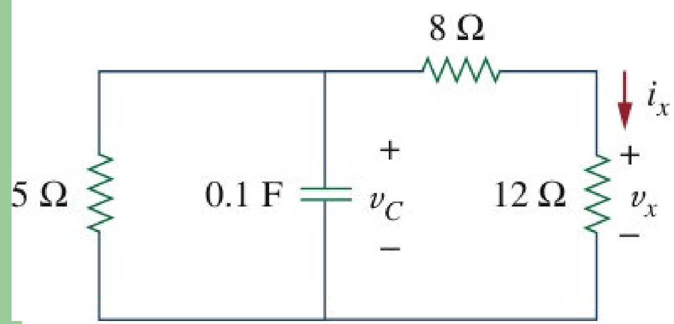
1. The initial voltage $v(0) = V_0$ across the capacitor
2. The time constant τ (τ=RC, R is usually the Thevenin equivalent resistance)

With these 2 items, we obtain the capacitor voltage $v_c(t)=v(t)=v(0)e^{-t/\tau}$.

Once v_c *is found,* i_c , v_R *and* i_R *can be determined*

Example

Let $v_c(0)$=15V. Find v_c,v_x and i_x for t>0



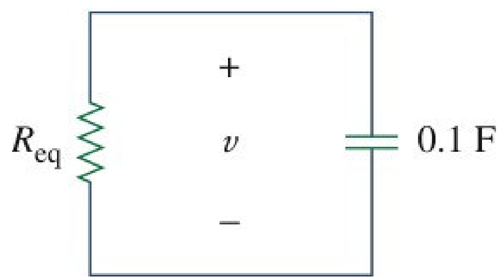
1st: Convert the circuit into standard RC circuit

2nd: Find the equivalent resistance or Thevenin resistance at the capacitor terminals

3rd: Goal is to find v_c 1st the solve v_x and i_x

8Ω and 12Ω are in series = 20Ω

20Ω and 5Ω are in parallel = $\frac{20 \times 5}{20+5}$ =4Ω



$$v(t) = V_0 e^{-t/\tau}$$

$$\tau = R_{eq}C$$

$$\tau = 4(0.1) = 0.4s$$

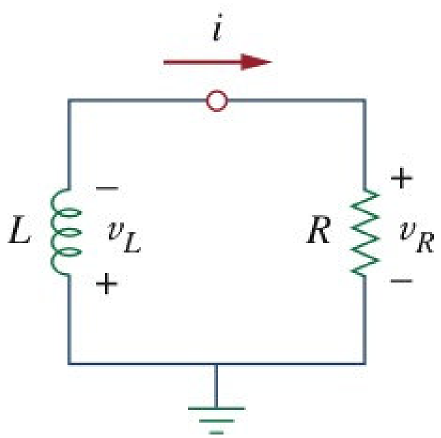
$$Thus, v = v(0)e^{-t/0.4} \quad v_c = v = 15e^{-2.5t}\text{V}$$

$$v_x = \frac{12}{12+8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\text{V}$$

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\text{A}$$

Source-Free *RL* Circuit

Goal is to determine the circuit response, which assume to be the current *i(t)* through the inductor.



At time $t = 0$, it is assumed that the inductor has an initial current I_0

$$i(0) = I_0$$

$$v_L + v_R = 0$$

$$L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

(1^{st} order differential eq)

Integrating both sides,

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}dt$$

$$\ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

$$\ln\frac{i(t)}{I_0} = -\frac{Rt}{L}$$

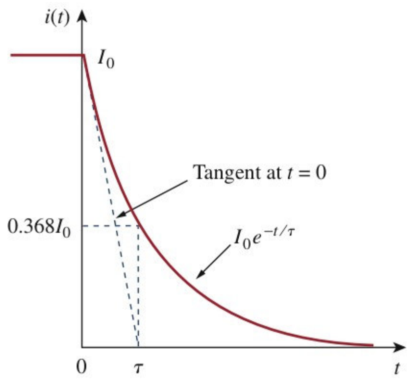
$$i(t) = I_0 e^{-Rt/L}$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current.

As t increases, the current decreases to zero and the rapidity with which the current decreases is expressed in term of the time constant (τ). It is evident that from above eq that the time constant for the RL circuit is

$$\tau = \frac{L}{R}$$

Cont…



$$i(t) = I_0 e^{-t/\tau}$$

Voltage across the resistor is

$$v_R(t) = iR = I_0 e^{-t/\tau} R$$

The power dissipated in the resistor is

$$p = v_R i = I_0^{\ 2} e^{-2t/\tau} R$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p\,dt = \int_0^t I_0^{\ 2} e^{-2t/\tau} R\,dt$$

$$= -\frac{1}{2}\tau I_0^{\ 2} e^{-2t/\tau}\Big|_0^t$$

$$= \frac{1}{2} L I_0^{\ 2}(1 - e^{-2t/\tau})$$

Note as $\mathrm{t} \to \infty, \mathrm{w_R}(\infty) \to \frac{1}{2} L I_0^{\ 2}$ which is the same the initial energy stored in th inductor.

The energy initially stored in the inductor is eventually dissipated in the resistor

Source-free RL circuit Summary

The Key to working with a source-free *RL* circuit is to find:

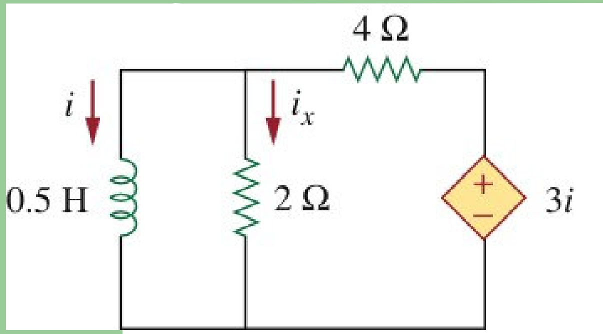
1. The initial current $i(0) = I_0$ through the inductor
2. The time constant τ ($\tau = L/R$, R is usually the Thevenin equivalent resistance)

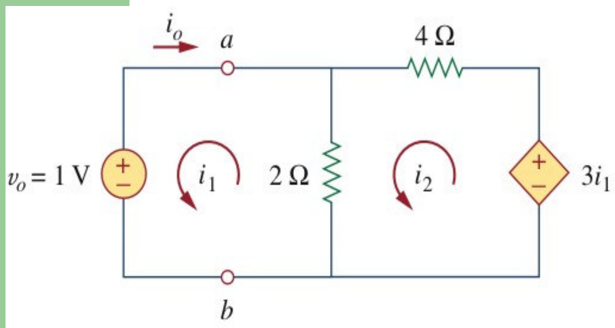
With these 2 items, we obtain the inductor current $i_L(t) = i(t) = i(0)e^{-t/\tau}$.

Once I_L *is found,* V_L , v_R *and* i_R *can be determined*

Example

Assuming that $i(0)$=10A, calculate $i(t)$ and $i_x(t)$ in the circuit





Method 1

1st: The equivalent resistance is the same as thevenin resistance at the inductor terminal

2nd: Add a voltage source with v_0=1V because of the dependent source

Applying KVL to the 2 loops :

$$2(i_1 - i_2) + 1 = 0 \qquad i_1 - i_2 = -\frac{1}{2} \text{........ } eq(1)$$

$$6i_2 - 2i_1 - 3i_1 = 0 \qquad i_2 = -\frac{5}{6} i_1 \text{........ } eq(2)$$

substitue eq(2) into eq(1)

$$i_1 = -3\text{A} \qquad i_0 = i_1 = 3\text{A}$$

$$\text{Hence} \qquad R_{eq} = R_{Th} = \frac{v_0}{i_0} = \frac{1}{3}\,\Omega$$

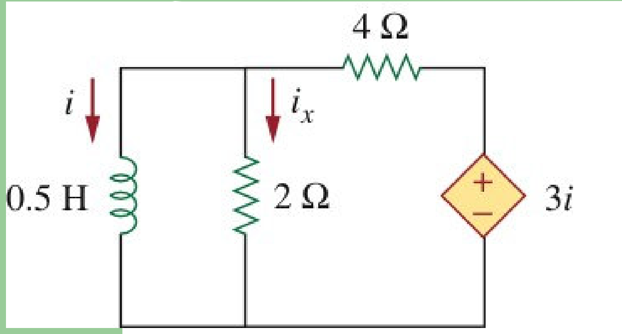
$$\text{Time constant,} \qquad \tau = \frac{L}{R_{eq}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\,s$$

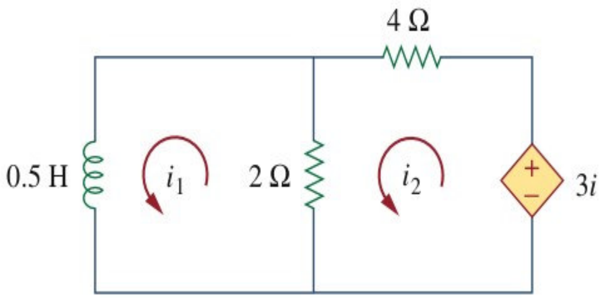
The current through the inductor is

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t}\,\text{A} \qquad t > 0$$

Example

Assuming that *i(0)*=10A, calculate *i(t)* and $i_x(t)$ in the circuit





Method 2

1st: Using KVL

For loop 1:

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0 \qquad or$$

$$\frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \qquad ......eq(1)$$

For loop 2:

$$6i_2\text{-}2i_1\text{-}3i_1 = 0, \quad i_2 = \frac{5}{6}i_1.....eq(2)$$

substitute eq (2) into eq(1)

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0$$

$$\frac{di_1}{dt} = -\frac{2}{3}i_1$$

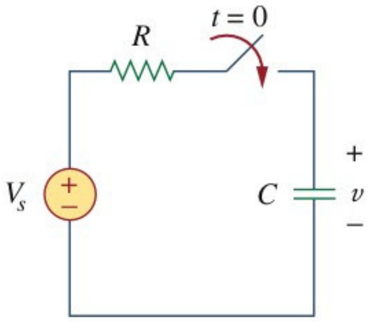
Since $i_1 = i$, replace i_1 with i and integrate :

$$\ln i\Big|_{i(o)}^{i(t)} = -\frac{2}{3}t\Big|_0^t \qquad \text{or } \ln\frac{i(t)}{i(0)} = -\frac{2}{3}t$$

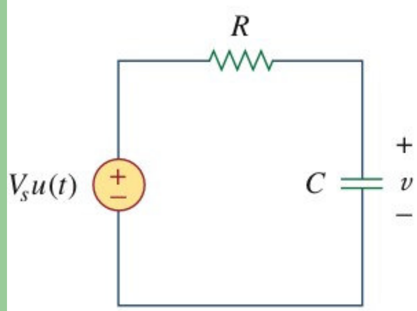
$$i(t) = i(0)e^{-(2/3)t} = \quad 10\, e^{-(2/3)t}\,\text{A}$$

Step Response of an *RC* Circuit

When the dc source of the RC circuit is suddenly applied, the voltage or current source can be modelled as a step function, and the response is known as a step function



Before switching



After switching

1st: Assume an initial voltage V_0 on the capacitor, this is not necessary for the step response

$$v(0^-) = v(0^+) = V_0$$

$v(0^-)$ is the voltage across capacitor just before switching
$v(0^+)$ is the voltage across capacitor just after switching

$$c\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0 \qquad \text{or}$$

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} u(t)$$

Where v is the voltage across the capacitor. For *t>0*

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

$$\frac{dv}{dt} = -\frac{v - V_s}{RC} \qquad \text{or}$$

$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

Cont…

Integrating both sides and introducing the initial conditions

$$\ln(v - Vs)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0 \qquad \text{or}$$

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau} \qquad \tau = RC$$

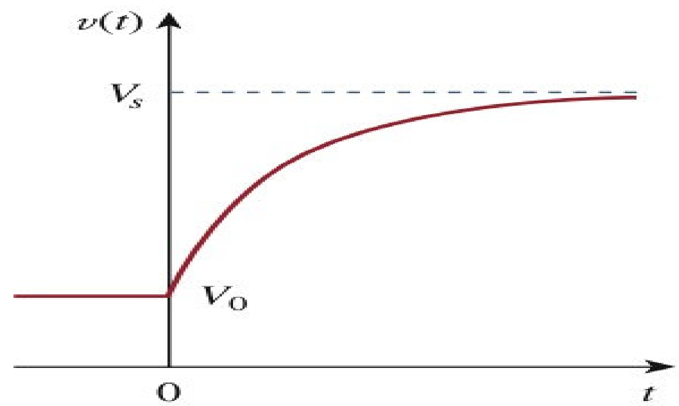
$$v - V_s = (V_0 - V_s)e^{-t/\tau} \quad \text{or}$$

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau} \qquad t > 0$$

$$v(t) = V_0 \qquad t < 0$$

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau} \qquad t > 0$$

This is known as the complete response (or total response) of the RC circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged



Cont…

If the capacitor is assume to be uncharged initially, $V_0=0$

$$v(t) = 0 \qquad t < 0$$

$$v(t) = V_s(1 - e^{-t/\tau}) \qquad t > 0$$

which can be written as

$$v(t) = V_s(1 - e^{-t/\tau})u(t) \qquad \text{or}$$

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t)$$

There is a short-cut method for finding the step response of an RC and RL circuit, let look at the equation below

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau} \longrightarrow$$ *v(t)* has 2 components

Complete response = natural response + forced response

stored energy independent source

$$v = v_n + v_f$$

$$v_n \downarrow V_0 e^{-t/\tau} \qquad v_f \downarrow V_s(1 - e^{-t/\tau})$$

Cont…

Another method is to break the complete response into 2 components:

Complete response = transient response + steady-state response

temporary part permanent part

$$v = v_t + v_{ss}$$

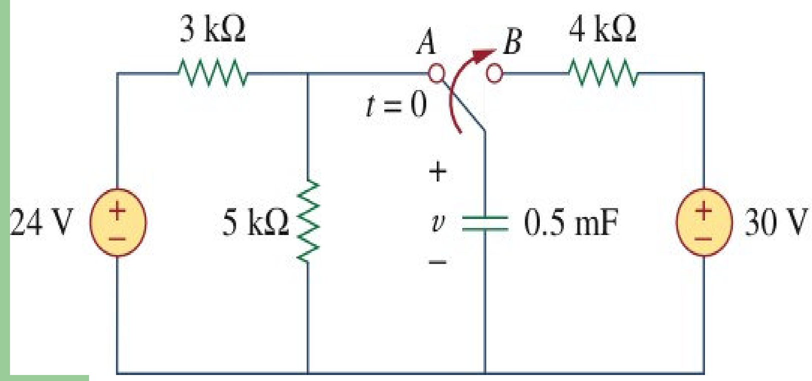
$$v_t \downarrow (v_0 - v_s)e^{-t/\tau} \qquad v_{ss} \downarrow v_s$$

Whichever way we look at it, the complete response may be written as

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

Example

The switch in the circuit below has been in position A for a long time. At *t=0*, the switch moves to B. Determine *v(t)* for *t>0* and calculate its value at *t=1*s and *4s*



For t<0, the switch is at position A. The capacitor acts like an open circuit to dc, but v is the same as voltage across the 5KΩ resistor. Hence, the voltage across the capacitor just before t=0 is obtained by divided as

$$v(0^-) = \frac{5}{5+3}(24) = 15\text{V}$$

Since the capacitor voltage can't change instantenously,

$$\text{v}(0) = \text{v}(0^-) = \text{v}(0^+) = 15\text{V}$$

For t > 0, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{Th} = 4K\Omega$, and the time constant is

$$\tau = R_{TH}C = 4x10^3 x0.5x10^{-3} = 2s$$

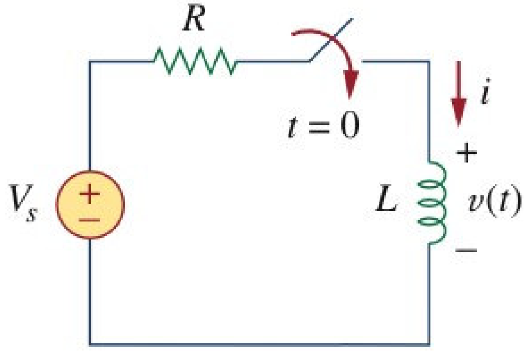
Since the capacitor acts like an open crcuit to dc at steady state, $v(\infty) = 30V$

$$\text{v(t)} = \text{v}(\infty) + [\text{v}(0) - \text{v}(\infty)]\text{e}^{-t/\tau}$$
$$= 30 + (15 - 30)\text{e}^{-t/2}$$
$$= (30 - 15\text{e}^{-0.5t})\text{V}$$

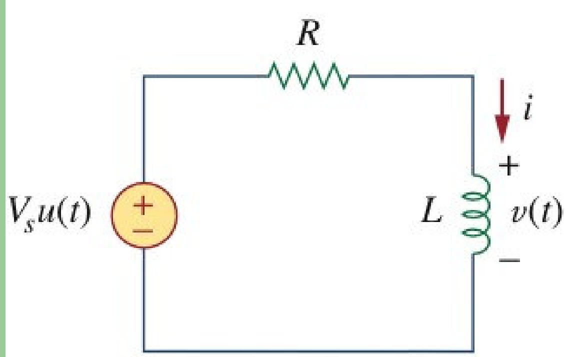
at t = 1, $\quad v(1) = 30 - 15e^{-0.5} = 20.9V$

at t = 4, $\quad v(4) = 30 - 15e^{-2} = 27.97V$

Step Response of an *RL* Circuit



Before switching



After switching

$$i = i_t + i_{ss}$$

$$i_t = Ae^{-t/\tau} \text{ (transient response)} \qquad \tau = \frac{L}{R}$$

$$i_{ss} = \frac{V_s}{R} \qquad \text{(steady - state response)}$$

$$i = Ae^{-t/\tau} + \frac{V_s}{R} \qquad \text{.......... .... .eq(1)}$$

Determine the constant A from the initial value of i. Let I_0 be the initial current through the inductor, which may come from a source other than V_s.

$$i(0^+) = i(0^-) = I_0$$

$$t = 0, \qquad \text{eq(1) becomes}$$

$$I_0 = A + \frac{V_s}{R}$$

$$A = I_0 - \frac{V_s}{R}$$

$$i(t) = \frac{V_s}{R} + (I_0 - \frac{V_s}{R})e^{-t/\tau}$$

This is the complete response of the *RL* Circuit

Cont…

The complete response can be written as

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

if the switching take place at time t = to instead of t = 0

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau}$$

$$If \;\; I_0 = 0$$

$$i(t) = 0 \qquad\qquad \mathrm{t} < 0$$

$$i(t) = \frac{\mathrm{V}_s}{R}(1 - \mathrm{e}^{-\mathrm{t}/\tau}) \qquad\qquad\qquad \mathrm{t} < 0$$

or

$$i(t) = \frac{\mathrm{V}_s}{R}(1 - \mathrm{e}^{-\mathrm{t}/\tau})u(t)$$

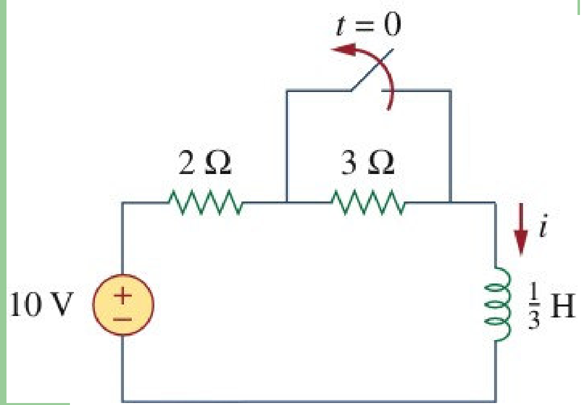
This is the step response of the RL circuit with no initial inductor current, the voltage across the inductor is

$$v(t) = L\frac{di}{dt} = Vs\frac{L}{\tau R}e^{-t/\tau}, \quad \tau = \frac{\mathrm{L}}{\mathrm{R}} \;\; \mathrm{t} > 0 \qquad \text{or}$$

$$v(t) = Vs\frac{L}{\tau R}e^{-t/\tau}u(t)$$

Example

Find i(t) in the circuit below for t>0. Assume that the switch has been closed for a long time.



When $t < 0$, the 3Ω resistor is short-circuited, and the inductor acts like short circuit. The current through the inductor at $t = 0^-$ (i.e., just before $t = 0$) is

$$i(0^-) = \frac{10}{2} = 5\text{A}$$

When $t > 0$, the switch is open. The 2Ω and 3Ω resistors are in series so that

$$i(\infty) = \frac{10}{2+3} = 2\text{A}$$

For The Thevenin resistance across the inductor terminal is

$$R_{TH} = 2 + 3 = 5\Omega$$

For the time constant

$$\tau = \frac{\text{L}}{\text{RTH}} = \frac{\frac{1}{3}}{5} = \frac{1}{15}s$$

Thus

$$\text{i(t)} = \text{i}(\infty) + [\text{i}(0) - \text{i}(\infty)]\text{e}^{-t/\tau}$$

$$= 2 + (5-2)\text{e}^{-15\text{t}} = 2 + 3e^{-15t}\text{A} \qquad \text{t} > 0$$

To check, for $t > 0$, KVL must be satified; that is

$$10 = 5\text{i} + \text{L}\frac{\text{di}}{\text{dt}}$$

$$5i + L\frac{di}{dt} = [10 + 15e^{-15t}] + [\frac{1}{3}(3)(-15)e^{-15t} = 10$$

VII-Second-Order Circuits

Introduction

In this section circuits containing 2 storage elements will be considered, this is also known as second-order circuit because their responses are described by differential equations that contain 2_{nd} derivatives. Typical 2_{nd}-order circuits are *RLC* circuits.

Finding Initial and Final Values

In this chapter *v* denoted capacitor voltage, while *i* is the inductor current. This section is explicitly devoted of getting $v(0), i(0), dv(0)/dt, di(0)/dt, i(\infty)$ ***and*** $v(\infty)$.

There are 2 key points in the determining the initial condition:

1st: Handle the polarity of voltage *v(t)* across the capacitor and the direction of the current *i(t)* through the inductor carefully

2nd: Keep in mind that the capacitor voltage is always continuous so that

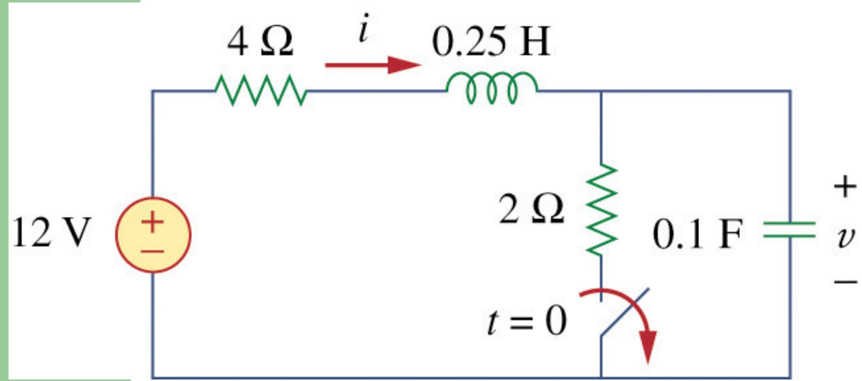
$$v(0^+) = v(0^-)$$

And the inductor current is always continuous so that

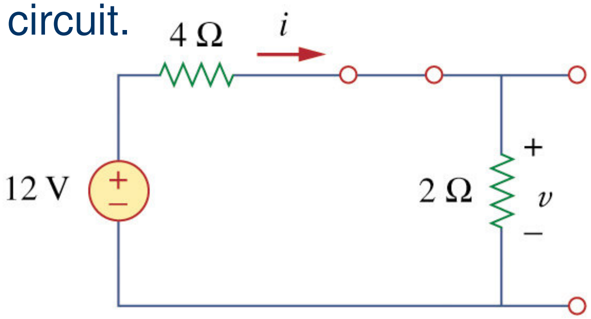
$$i(0^+) = i(0^-)$$

Example

The switch in the circuit below has been closed for a long time. It is open at t=0. Find (a) $i(0^+), v(0^+)$,(b) $di(0^+)/dt, dv(0^+)/dt$,(c) $i(\infty), v(\infty)$



The switch is closed for a long time before t=0. At t=0 (dc steady state), the inductor acts like a short circuit, while the capacitor acts like an open circuit.



At $t=0^-$,

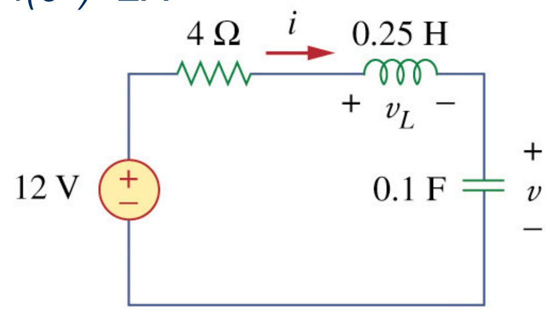
$i(0^-)$= 12/(4+2)=2A, $v(0^-)=2i(0^-)$=4V

As the inductor current and the capacitor voltage cannot change abruptly,

$i(0^+)=i(0^-)$=2A, $v(0^+)=v(0^-)$=4V

At $t=0^+$, the switch is open;The same current flows through both the inductor and and capacitor. Hence

$i_c(0^+)=i(0^+)$=2A

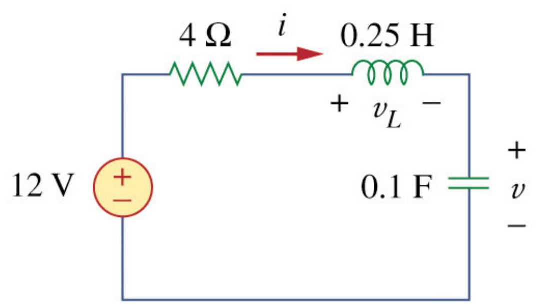


Since $Cdv/dt=i_C$, $dv/dt=i_c/C$ *and*

$dv(0^+)/dt=i_c(0^+)/C=2/0.1=20V/s$

Cont…

Similarly, since Ldi/dt=v_L/L. v_L can be obtained by applying KVL to the loop in the circuit below.



(b)

$-12+4i(0^+)+v_L(0^+)+v(0^+)=0$

$-12+4(2)+v_L(0^+)+4=0$

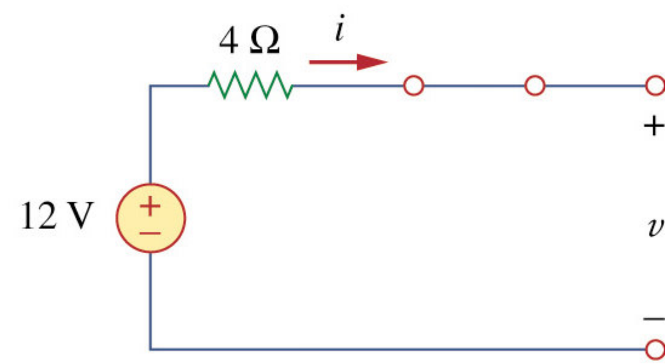
or

$v_L(0+)=12-8-4=0V$

Thus,

$di(0^+)/dt=v_L(0^+)/L=0/0.25=0A/s$

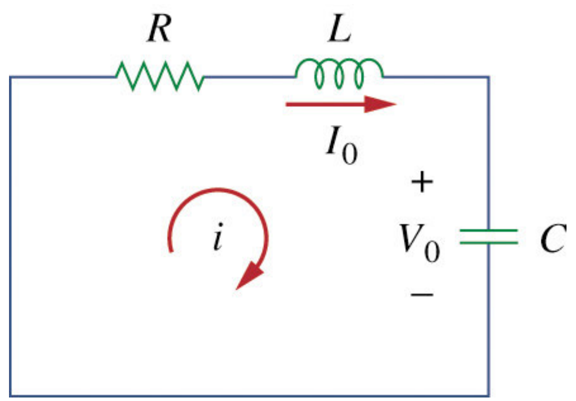
For t>0, the circuit undergoes transience. But as t $\longrightarrow \infty$, the circuit reaches steady state again. The inductor acts like a short circuit and the capacitor like an open circuit , so that the circuit becomes



$i(\infty)=0A$, $v(\infty)=12V$

Source-free Series *RLC* Circuit

It is important to understand the natural response of the series *RLC* circuit.



The circuit being excited by energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage v_0 and initial inductor current I_0. Thus at t=0

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i dt = V_0 \text{ initial condition}$$

$$i(0) = I_0 \text{ initial condition}$$

Applying KVL around the loop

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i dt = 0$$

$$\frac{Ri}{L} + \frac{L}{L}\frac{di}{dt} + \frac{1}{LC}\int_{-\infty}^{t} i dt = 0$$

$$\frac{R}{L}\frac{di}{dt} + \frac{d^2 i}{dt^2} + \frac{i}{LC} = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \text{.........eq(1)}$$

This is a 2nd-order differential equation. In order to solve the equation, it requires two initial conditions

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

$$L\frac{di(0)}{dt} + RI_0 + V_0 = 0$$

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) = 0 \text{....used to solve eq(1)}$$

Cont...

Eq(1) can be solved using exponential form

$$i = Ae^{st}$$

A and s are constants, $i = Ae^{st}$ int o eq(1)

$$\frac{d^2(Ae^{st})}{dt^2} + \frac{ARd(e^{st})}{Ldt} + \frac{Ae^{st}}{LC} = 0$$

$$As^2e^{st} + \frac{As\,\mathrm{R}e^{st}}{L} + \frac{Ae^{st}}{LC} = 0$$

$$Ae^{st}(s^2 + \frac{R}{L}s + \frac{1}{LC}) = 0$$

i=Ae^{st} is the assumed solution, so only solve the expression in parentheses

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

The quadratic equation is known as the characteristic equation of the differential eq (1), since the roots of the equation dictate the character of i.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

A more compact wa y of expressing the roots is

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^{\ 2}}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^{\ 2}},$$

where

$$\alpha = \frac{\mathrm{R}}{2L} \qquad \omega_0 = \frac{1}{\sqrt{\mathrm{LC}}}$$

where a=R/2L ω_0=1/LCThe roots s_1 and s_2 are called natural frequencies, ω_0 is known as the resonant frequency or undamped natural frequency, α is the neper frequency or the damping factor

$$s^2 + 2\alpha s + \omega_0^{\ 2} = 0$$

Cont…

$$s^2 + 2\alpha s + \omega_0^{\,2} = 0$$

s and ω_0^2 are important quantities that will be discuss throughout the rest of the section

The 2 values of s in the above equation indicate there are 2 possible solutions for *i,* each of which is the form of assumed solution for equation

$$i_1 = A_1 e^{s_1 t}, \qquad i_2 = A_2 e^{s_2 t}$$

Since eq(1) is a linear equation, any linear combination of the 2 distinct solution of i_1 and i_2 is also a solution of eq(1)

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0\text{.........eq(1)}$$

The natural response of the series of the series RLC circuit is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where the constants A_1 andA_2 are determined from the initial values i(0) and di(0)/dt

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^{\,2}}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^{\,2}},$$

From the equation above, we can infer 3 types of solutions:

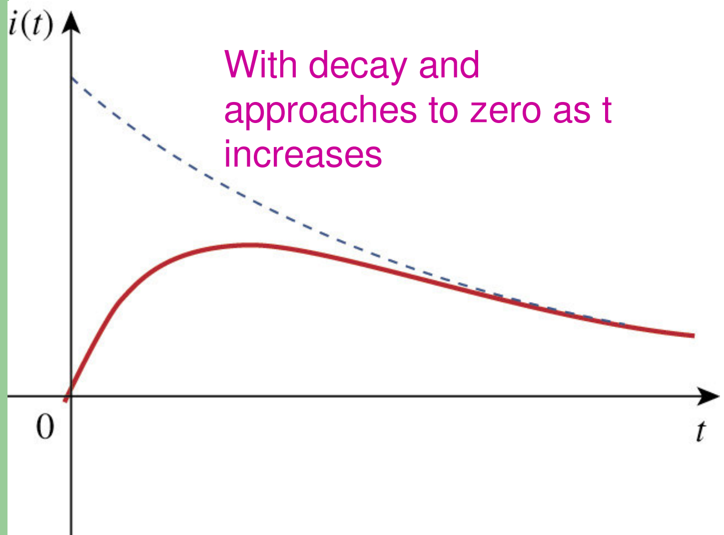
1.If $\alpha > \omega_0$, we have the overdamped case

2.If $\alpha = \omega_0$, we have the critically damped case

3.If $\alpha < \omega_0$, we have the underdamped

Each case will be considered separately

Overdamped Case ($\alpha > \omega_0$) implies $C > 4L/R^2$

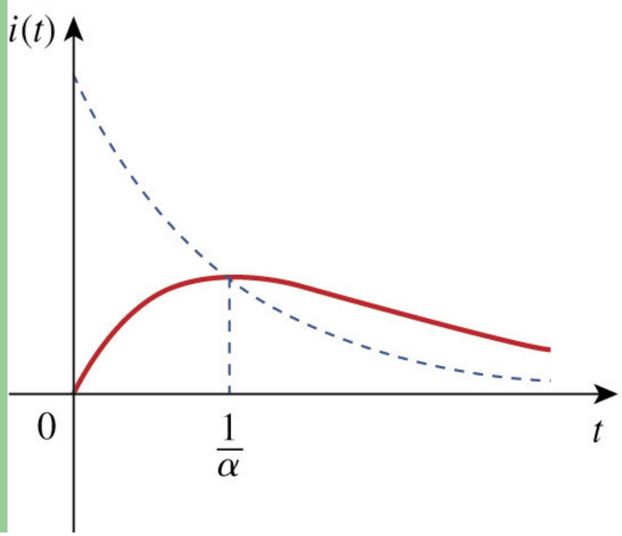


With decay and approaches to zero as t increases

s_1 and s_2 are real and negative. The response is as followed

$$i_1 = A_1 e^{s_1 t}, \qquad i_2 = A_2 e^{s_2 t}$$

Critical Damped Case ($\alpha = \omega_0$) implies $C = 4L/R^2$



$s_1 = s_2 = -\alpha = R/2L$

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

$A_3 = A_1 + A_2$. This cannot be the solution, because the 2 initial conditions cannot be satisfied with the single constant A_3.

Cont…

Let go back to eq(1).
When $\alpha=\omega_0= R/2L$ eq(1) becomes

$$\frac{d^2 i}{dt^2}+2\alpha\frac{di}{dt}+\alpha^2 i=0$$

$$\frac{d}{dt}\left(\frac{di}{dt}+\alpha i\right)+\left(\frac{di}{dt}+\alpha i\right)=0......eq(a)$$

$$\text{Let } f=\frac{di}{dt}+\alpha i..........eq(b)$$

substitute eq(b) into eq(a)

$$\frac{df}{dt}+\alpha f=0$$

the above equation is 1st-order differential equation with solution $f=A_1 e^{-\alpha t}$, where A_1 is a constant

$$\frac{di}{dt}+\alpha i=A_1 e^{-\alpha t}$$

$$e^{\alpha t}\frac{di}{dt}+e^{\alpha t}\alpha i=A_1$$

$$\frac{d}{dt}e^{\alpha t}=i\frac{d}{dt}e^{\alpha t}+e^{\alpha t}\frac{d}{dt}i$$

$$\frac{d}{dt}(e^{\alpha t} i)=A_1$$

Integrating both side

$$e^{\alpha t} i=A_1 t+A_2 \text{ or } i=(A_1 t+A_2)e^{-\alpha t}$$

A2 is a constant. The natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term, or

$$i=(A_1 t+A_2)e^{-\alpha t}$$

Cont…

Underdamped Case ($\alpha<\omega_0$) implies $C<4L/R^2$

$$s_1 = -\alpha + \sqrt{-(\omega_0{}^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{-(\omega_0{}^2 - \alpha^2)} = -\alpha - j\omega_d$$

$\mathrm{j} = \sqrt{-1}$ and $\omega_\mathrm{d} = \sqrt{\omega_0{}^2 - \alpha^2}$ which is called the damping frequency.

ω_0=undamped natural response, ω_d=damped natural frequency

$$i(t) = A_1 e^{-(\alpha - jw_d)t} + A_2 e^{-(\alpha + jw_d)t}$$

$$= e^{-\alpha t}(A_1 e^{jw_d t} + A_2 e^{-jw_d t})$$

Using Euler' s identities ,

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad e^{-j\theta} = \cos\theta - j\sin\theta$$

$$i(t) = e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)]$$

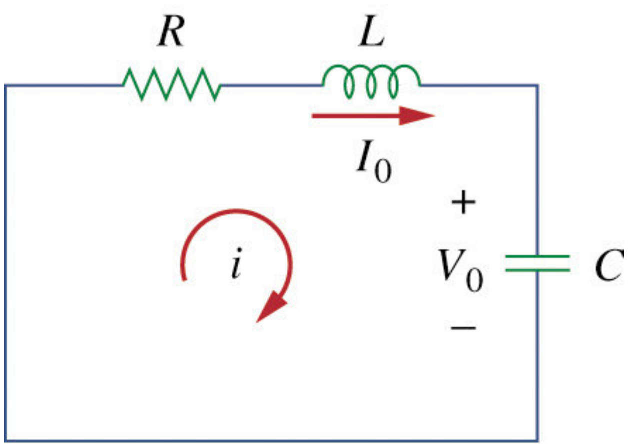
$$= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t]$$

Replacing constant (A_1+A_2) and $j(A_1-A_2)$ with constant B_1 and B_2

$$i(t) = e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t)$$

Example

In below circuit, R=40Ω, L=4H and C=1/4F. Calculate the characteristics roots of the circuit. Is the natural response overdamped, underdamped or critically damped



$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4x\frac{1}{4}}} = 1$$

$$\alpha > \omega_0$$

The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - w_o{}^2} = -5 \pm \sqrt{25-1}$$

$$s_1 = -0.101$$

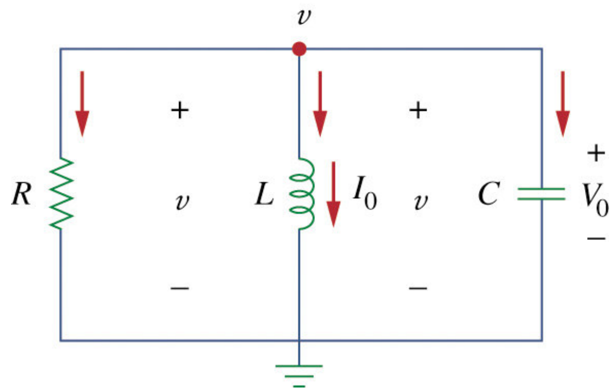
$$s_2 = -9.899$$

$\alpha > \omega_0$, is overdamped

The roots are real and negative

Source-free Parallel *RLC* Circuit

Look at the parallel *RLC* circuit below, assume initial inductor current I_0 and initial capacitor voltage V_0



$$i(0) = I_o = \frac{1}{L}\int_{\infty}^{0} v(t)dt$$

$$v(0) = V_0$$

Since the 3 elements are in parallel, they have the same voltage v across them

Apply KCL at the top node,

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} vdt + C\frac{dv}{dt} = 0$$

Taking the derivative with respect to t and dividing by C

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

similarly

$$s^2 + \frac{1}{RC}s + \frac{1}{LC}v = 0$$

The roots of the characteri stics equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \qquad w_o = \frac{1}{\sqrt{LC}}$$

Cont…

There are 3 possible solutions depending $\alpha > \omega_0$ (overdamped), $\alpha = \omega_0$ (critically damped) or $\alpha < \omega_0$ (underdamped)

Overdamped Case ($\alpha > \omega_0$) when $L > 4R^2C$.

The roots of the characteristics equation are real and negative. The response is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Critically Damped Case ($\alpha = \omega_0$) when $L = 4R^2C$.

The roots are real and equal so that the response is

$$v(t) = (A_1 + A_2) e^{-\alpha t}$$

Underdamped Case ($\alpha < \omega_0$) when $L < 4R^2C$.

The roots are complex and may be expressed as

$$s_{1,2} = -\alpha \pm j\omega_d \qquad \text{where } \omega_d = \sqrt{\omega o^2 - \alpha^2}$$

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

Cont…

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

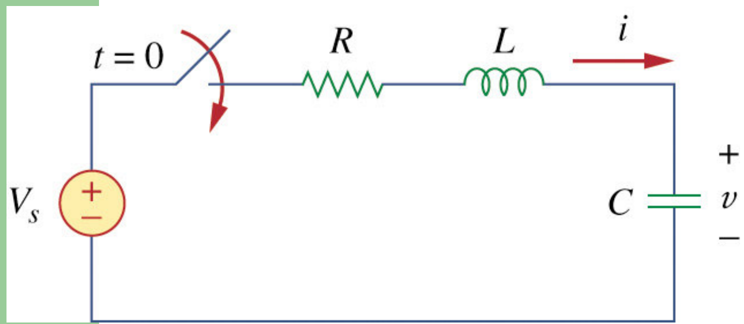
The constant A_1 and A_2 in each case can be determined from initial conditions, we need v(0) and dv(0)/dt

$$\frac{V_o}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}$$

Step Response of a Series RLC Circuit

A step response is obtained by the sudden application of a dc source.



Applying KVL around the loop for t>0

$$L\frac{di}{dt}+Ri+v=V_s \qquad \text{but } i=C\frac{dv}{dt}$$

$$LC\frac{d^2v}{dt^2}+RC\frac{dv}{dt}+v=V_s$$

$$\frac{d^2v}{dt^2}+\frac{RC}{LC}\frac{dv}{dt}+\frac{v}{LC}=\frac{V_s}{LC}$$

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{v}{LC}=\frac{V_s}{LC} \text{......eq(2)}$$

Same as eq(1) from early note

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{LC}=0\text{.........eq(1)}$$

The characteristic equation for the series RLC circuit is not affected by the presence of the dc source.

Equation (2) has 2 components:

The transients response $v_t(t)$ and steady state response $v_{ss}(t)$: $v(t)=v_t(t)+v_{ss}(t)$

$v_t(t)$=total response that dies out with time. Transient response is the same as the form of the solution obtained in source free series RLC circuit.

$v_t(t)=A_1e^{s_1t}+A_2e^{s_2t}$ (overdamped)

$v(t)=(A_1+A_2)e^{-\alpha t}$ (critically damped)

$v(t)=(A_1\cos\omega_d t+A_2\sin\omega_d t)e^{-\alpha t}$ (underdamped)

Cont…

The steady-state response is the final value of v(t). The final value of the capacitor voltage is the same as the source voltage V_s. Hence, $v_{ss}(t)=v(\infty)=V_s$

The complete solution for the overdamped,underdamped and critically damped cases are

$v_t(t)=V_S+A_1e^{s_1t}+A_2e^{s_2t}$ (overdamped)

$v(t)=V_s+(A_1+A_2)e^{-\alpha t}$ (critically damped)

$v(t)=V_s+(A_1\cos\omega_dt+A_2\sin\omega_dt)e^{-\alpha t}$ (underdamped)

The values of the constants A_1 and A_2 are obtained from the initial condition:v(0) and dv(0)/dt

Reminder:

Once the capacitor voltage $v_c=v$ is known, i=Cdv/dt, which is the same current through the capacitor, inductor and resistor can be found.

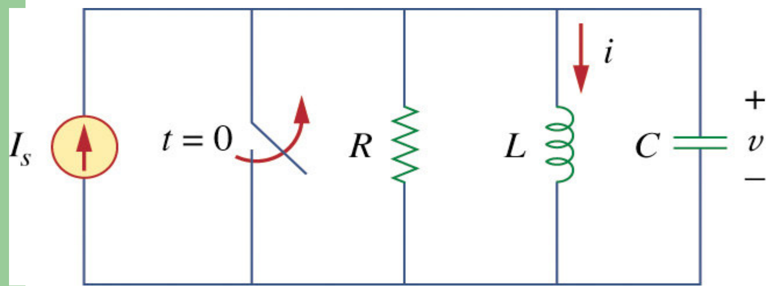
The complete response for any variable x(t) can be found directly, because it has the general form

$x(t)=x_{ss}(t)+x_t(t)$

$x_{ss}=x(\infty)$ is the final value and $x_t(t)$ is the transient response

Step Response of a Parallel RLC Circuit

Find I due to a sudden application of a dc current. Applying KCL at the top node for t>0



$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0.....eq(3)$$

Equation (4) has 2 components:

The transients response $i_t(t)$ and steady state response $i_{ss}(t)$: $i(t)=i_t(t)+i_{ss}(t)$

$$\frac{v}{R}+i+C\frac{dv}{dt}=I_s \qquad \text{but } v=L\frac{di}{dt}$$

$$\frac{L}{R}\frac{di}{dt}+i+LC\frac{d^2i}{dt^2}=I_s$$

$$LC\frac{d^2i}{dt^2}+\frac{L}{R}\frac{di}{dt}+i=I_s$$

$$\frac{d^2i}{dt^2}+\frac{L}{LCR}\frac{di}{dt}+\frac{i}{LC}=\frac{I_s}{LC}$$

$$\frac{d^2i}{dt^2}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{LC}=\frac{I_s}{LC}......eq(4)$$

Same as eq(3) from early note

The complete solution for the overdamped,underdamped and critically damped cases are

$i(t)=I_S+A_1e^{s_1t}+A_2e^{s_2t}$ (overdamped)

$i(t)=I_s+(A_1+A_2)e^{-\alpha t}$ (critically damped)

$i(t)=I_s+(A_1\cos\omega_d t+A_2\sin\omega_d t)e^{-\alpha t}$ (underdamped)

Cont…

The values of the constants A_1 and A_2 are obtained from the initial condition:*i* and di/dt

Reminder:

Once the inductor current i_L=i is known v=Ldi/dt, which is the same voltage across inductor capacitor, and resistor can be found.

The complete response for any variable x(t) can be found directly, because it has the general form

$x(t)=x_{ss}(t)+x_t(t)$

$x_{ss}=x(\infty)$ is the final value and $x_t(t)$ is the transient response

General Second-Order Circuit

Given a 2^{nd}-order circuit, we determine its step response $x(t)$(which may be voltage or current) by taking the following four steps:

1. Determine the initial condition $x(0)$ and $dx(0)/dt$ and the final value $x(\infty)$
2. Turn off the independent sources and find the form of the transient response $x_t(t)$by applying KCL and KVL. Once a 2^{nd}-order differential equation is obtained, determine its characteristics roots. Depending on whether the response is overdamped, critically damped or underdamped, $x_t(t)$ with two unknown constants will be obtained
3. Obtained the steady-state response as $x_{ss}(t)=x(\infty)$ (final value)
4. The total response is now found as the sum of the transient response and steady-state response: $x(t)=x_t(t)+x_{ss}(t)$

   Determine the constants associated with the transient response by imposing the initial condition $x(0)$ and $dx(0)/dt$ determined in step1

AC Circuit Sinusiods and Phasors

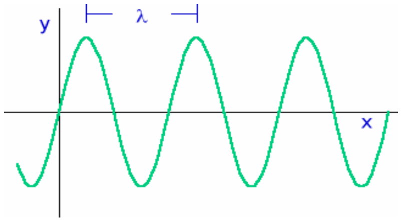
Introduction

A sinusoidal current is usually referred to as an alternating current (ac). Circuits driven by sinusoidal current or voltage are called ac circuits. Sinusoidal ac voltages are available from a variety of sources.

Wind-power station

Power Station

Hydroelectric Generators



Solar Panel

Oscilloscope

Sinusoids

Sinusoidal voltage: $v(t) = V_m \sin\omega t$

where

V_m = the amplitude of the sinusoid

ω = the angular frequency in radians/s

ωt = the argument of sinusoid

The time required to complete one revolution is equal to the period (T) of the sinusoidal waveform and the radians subtended in this time is are 2π

ω= the angular frequency in radians/s

$= 2\pi/T$ (rad/s)

v(t) repeats itself every T seconds.

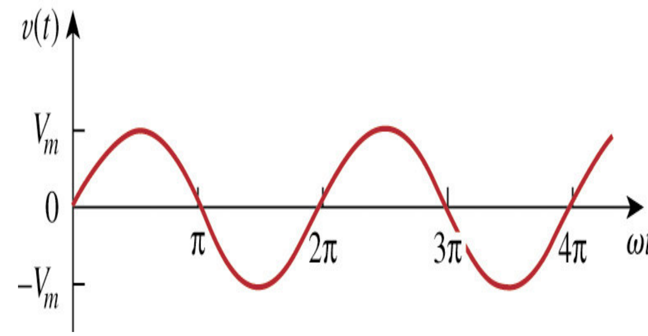
$v(t+T) = V_m \sin\omega(t+T) = V_m \sin\omega(t+2\pi/\omega)$

$= V_m \sin(\omega t+2\pi) = V_m \sin\omega t = v(t)$

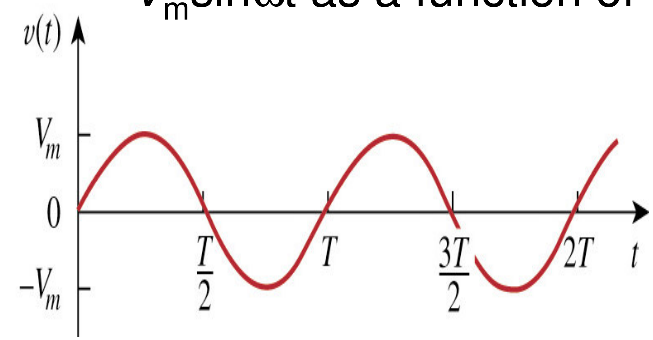
v(t+T) = v(t) is said to be periodic

A periodic function is one that satisfied f(t)=f(t+nT),

For all t and for all integers n



$V_m \sin\omega t$ as a function of ωt



$V_m \sin\omega t$ as a function of t

These are called sine waves. A sine wave is a *uniform wave that is generated by a single frequency*

$$f(t)=f(t+T)$$

The period T of the periodic function is the time of one complete cycle or the number of seconds per cycle. The reciprocal of this quantity is the number of cycles per seconds, know as the cyclic frequency f of the sinusoid.

$$f = 1/T$$

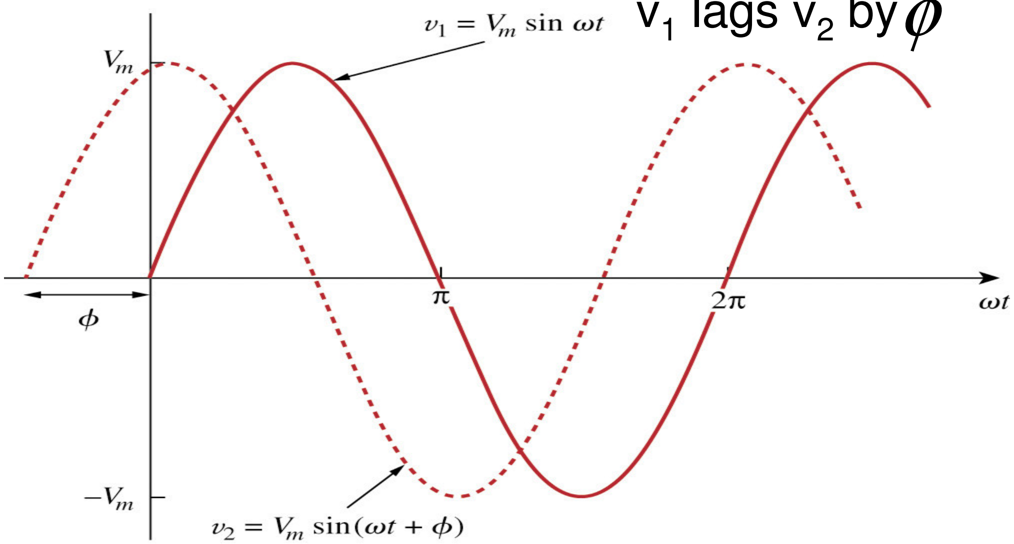
$$\omega = 2\pi f$$

ω is the radians per second (rad/s)
f is in hertz (Hz)

A sinusoid can be expressed in either sine or cosine form.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$



v_1 lags v_2 by ϕ

v_2 leads v_1 by ϕ

$$\sin(\omega t \pm 180^0) = -\sin \omega t$$

$$\cos(\omega t \pm 180^0) = -\cos \omega t$$

$$\sin(\omega t \pm 90^0) = \pm \cos \omega t$$

$$\cos(\omega t \pm 90^0) = \mp \sin \omega t$$

Cont…

Question

Calculate the phase angle between $v_1 = -10\cos(\omega t + 50^0)$ and $v_2 = 12\sin(\omega t - 10^0)$. State which sinusoid is leading.

Method 1

$$v_1 = -10\cos(\omega t + 50^0)$$
$$= 10\cos(\omega t + 50^0 - 180^0)$$
$$= 10\cos(\omega t - 130^0) \quad \text{or}$$
$$= 10\cos(\omega t + 230^0)$$
$$v_2 = 12\sin(\omega t - 10^0)$$
$$= 12\cos(\omega t - 10^0 - 90^0)$$
$$= 12\cos(\omega t - 100^0)$$

The phase difference between v_1 and v_2 is 30^0

$$v_2 = 10\cos(\omega t - 130^0 + 30^0) \quad \text{or}$$
$$v_2 = 10\cos(\omega t + 260^0)$$

v_2 leads v_1 by 30^0

Method 2

$$v_1 = -10\cos(\omega t + 50^0)$$
$$= 10\sin(\omega t + 50^0 - 90^0)$$
$$= 10\sin(\omega t - 40^0)$$
$$= 10\sin(\omega t - 10^0 - 30^0)$$
$$v_2 = 12\sin(\omega t - 10^0).$$

v_1 lags v_2 by 30^0

Phasor

A phasor is a complex number that represents both the magnitude and the phase of sinusoid.

A complex number z can be written in rectangular form as

$$z = x + jy \quad \text{(Rectangular form)}$$

x = real part of z
y = imaginary part of z

$$j = \sqrt{-1}$$

The complex number z can also be written in polar or exponential form as

$$z = r\angle\phi \qquad \text{(Polar form)}$$

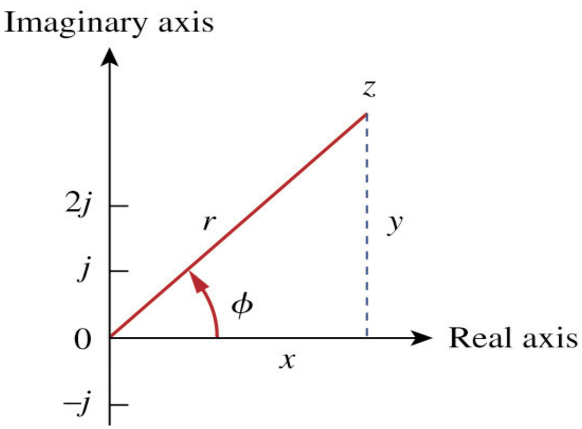
$$z = re^{j\phi} \qquad \text{(Exponential form)}$$

$$r = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi \qquad y = r\sin\phi$$

$$z = x + jy = r\angle\phi$$

$$z_1 = x_1 + jy_1 = r_1\angle\phi_1 \qquad z_2 = x_2 + jy_2 = r_2\angle\phi_2$$



Cont…

Given the complex number

$$z_1 = x_1 + jy_1 = r_1\angle\phi_1 \qquad z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

the following operations are important

$$Addition\ : z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$Subtraction\ : z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

$$Multiplication\ : z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$$

$$Division\ : \frac{z_1}{z_2} = \frac{r_1}{r_2}\angle \phi_1 - \phi_2$$

$$\mathrm{Re}\ ciprocal:\ \frac{1}{z} = \frac{1}{r}\angle - \phi$$

$$Square\ \ Root\ : \sqrt{z} = \sqrt{r}\angle \phi/2$$

$$Complex\ \ Conjugate\ : z^* = x - jy = r\angle - \phi = re^{-j\phi}$$

Cont…

The idea of phasor representation is based on Euler's Identity. In general,

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

$$\cos\phi = \mathrm{Re}(e^{j\phi})$$

$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

$$\text{Given a sinusoid } v(t) = V_m \cos(\omega t + \phi)$$

$$= \mathrm{Re}(V_m e^{j(\omega t + \phi)})$$

$$= \mathrm{Re}(V_m e^{j\phi} e^{j\omega t})$$

$$v(t) = \mathrm{Re}(V e^{j\omega t})$$

$$\mathrm{V} = \mathrm{V_m} e^{j\phi} = V_m \angle \phi,$$

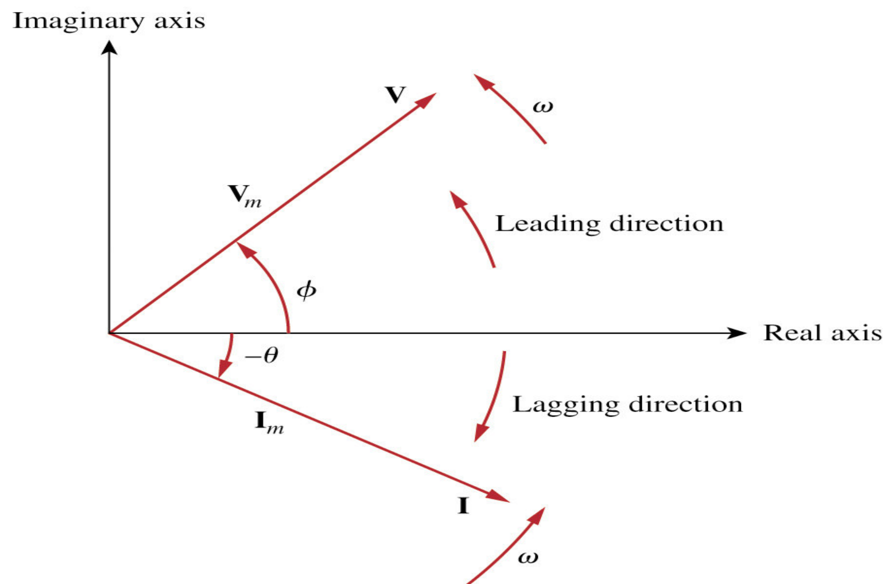
V is the phasor of the sinusoid v(t)

$$\mathrm{v(t)} = \mathrm{V_m}\cos(\omega t + \phi) \qquad \text{Time - domain}$$

$$\mathrm{V} = \mathrm{V_m}\angle\phi \qquad \text{Phasor - domain}$$

Cont…

A phasor diagram showing $\mathrm{V} = V_m \angle \phi$ and $\mathrm{I} = I_m \angle \phi$



Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^0$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^0$

Question

Evaluate these complex numbers :

a) $(40\angle 50^0 + 20\angle -30^0)^{1/2}$

Transform these sinusoids to phasors :

b) $\mathrm{i} = 6\cos(50\mathrm{t} - 40^0)\mathrm{A}$

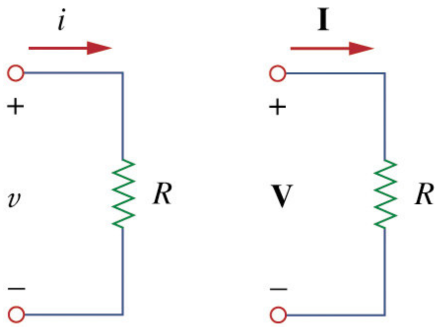
c) $\mathrm{v} = -4\sin(30\mathrm{t} + 50^0)\mathrm{V}$

d) Given $\mathrm{i}_1(t) = 4\cos(\omega t + 30^0)\mathrm{A}$ and $\mathrm{i}_2(t) = 5\sin(\omega t - 20^0)\mathrm{A}$, find their sum.

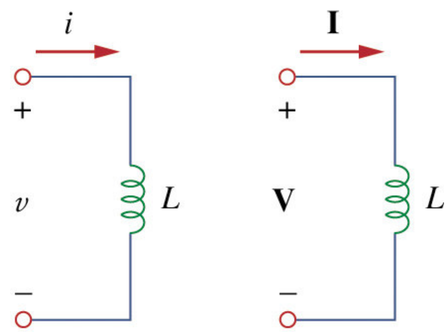
Phasor Relationship for Circuit Elements

How to represent a voltage or current in the phasor or frequency domain involving the passive elements R, L and C.

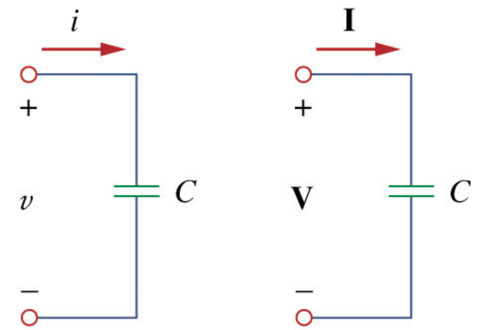
1st: Transform the voltage-current relationship from the time domain to the frequency domain for each element.



$v = iR$ (a) $\quad$ $\mathbf{V} = \mathbf{I}R$ (b)



$v = L\dfrac{di}{dt}$ (a) $\quad$ $\mathbf{V} = j\omega L\mathbf{I}$ (b)



$i = C\dfrac{dv}{dt}$ (a) $\quad$ $\mathbf{I} = j\omega C\mathbf{V}$ (b)

$$Current: i = I_m \cos(\omega t + \phi)$$

$$Voltage: v = iR$$

$$= RI_m \cos(\omega t + \phi)$$

Phasor Form

$$\mathrm{V} = RI_m \angle \phi$$

$$\mathrm{I} = I_m \angle \phi$$

$$Current \;: i = I_m \cos(\omega t + \phi)$$

$$Voltage \;: v = L\frac{di}{dt}$$

$$= -\omega L I_m \sin(\omega t + \phi)$$

$$\text{recall } -\sin \mathrm{A} = \cos(A + 90^0)$$

$$= \omega L I_m \cos(\omega \mathrm{t} + \phi + 90^0)$$

$$\mathrm{V} = \omega L I_m e^{j(\phi + 90^0)} = \omega L I_m e^{j\phi} e^{90^0}$$

$$= \omega L I_m \angle \phi + 90^0$$

$$= j\omega L \mathrm{I}$$

$$Voltage: v = v_m \cos(\omega t + \phi)$$

$$Current: i = c\frac{dv}{dt}$$

$$\mathrm{I} = j\omega C V$$

$$\mathrm{V} = \frac{\mathrm{I}}{j\omega C}$$

Question

The voltage $v = 12\cos(60t + 45^{0})$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor

Impedance

The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms (Ω)

$$\mathrm{V} = R\mathrm{I} \qquad \mathrm{V} = j\omega L\mathrm{I} \qquad \mathrm{V} = \frac{1}{j\omega C}$$

$$\mathrm{Z} = \frac{\mathrm{V}}{\mathrm{I}} = R \qquad \mathrm{Z} = \frac{\mathrm{V}}{\mathrm{I}} = j\omega L \qquad \mathrm{Z} = \frac{\mathrm{V}}{\mathrm{I}} = \frac{1}{j\omega C}$$

when $\omega = 0$

i.e., in dc source : $\mathrm{Z_L} = 0$ (inductor acts as short circuit)

and $\mathrm{Z_C} \to \infty$ (capacitor acts as open circuit)

when $\omega \to \infty$

i.e., for high frequency : $\mathrm{Z_L} \to \infty$ (inductor acts as open circuit)

and $\mathrm{Z_C} = 0$ (capacitor acts as short circuit)

As a complex quantity, the impedance may be expressed in rectangular form as

$$Z = R + jX = |Z| \angle \theta$$

$$|Z| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1} \frac{X}{R}$$

$$R = |Z| \cos\theta, \qquad \mathrm{X} = |\mathrm{Z}| \sin\theta$$

$R = \mathrm{Re}\, Z$ is the resistance

$\mathrm{X} = \mathrm{Im}\,\mathrm{Z}$ is the reactance

Admittance

The admittance y is the reciprocal of impedance, measured in siemens (S)

$$\mathrm{Y} = \frac{1}{\mathrm{R}} \qquad \mathrm{Y} = \frac{1}{j\omega L} \qquad \mathrm{Y} = j\omega C$$

As a complex quantity, the admittance may be expressed in rectangular form as

$$Y = G + jB$$

$$G + jB = \frac{1}{R + jX}$$

$$= \frac{1}{R + jX} \bullet \frac{R - jX}{R - jX}$$

$$= \frac{R - jX}{R^2 + X^2}$$

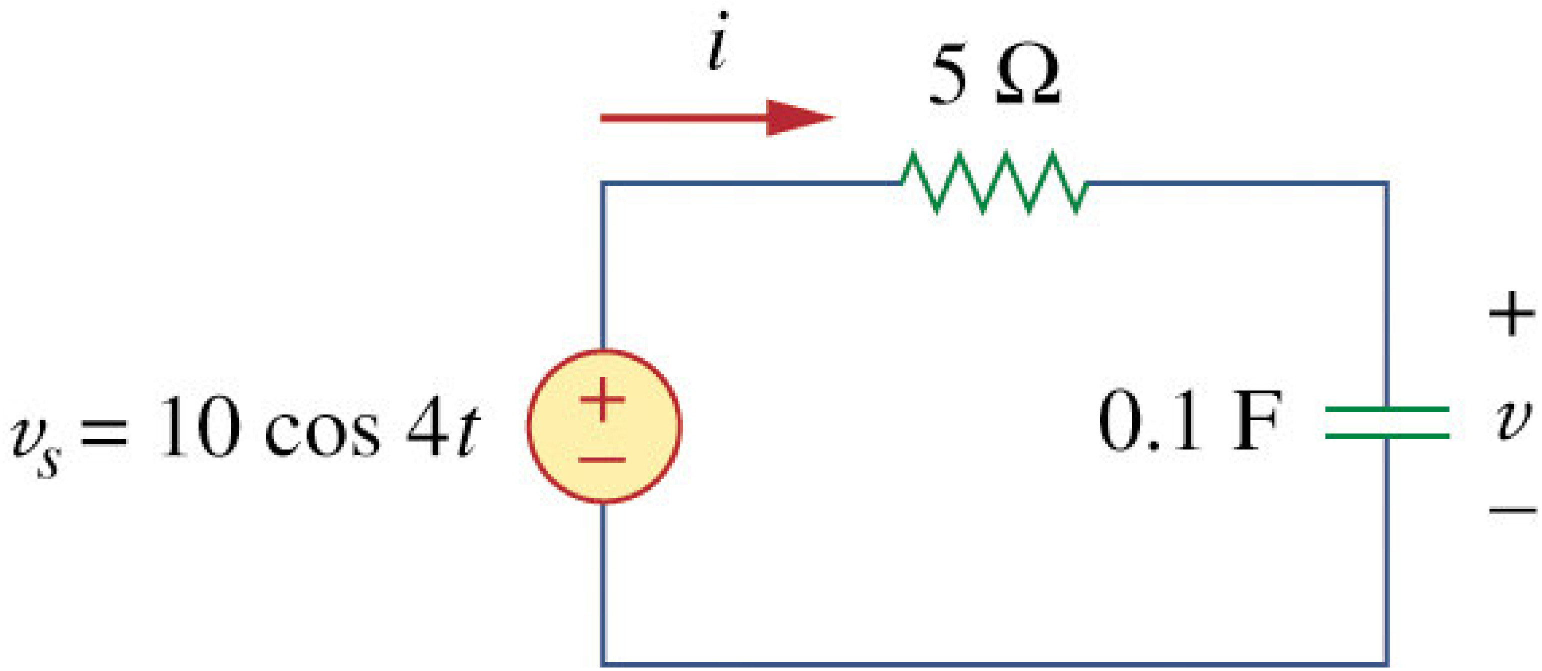
$G = \mathrm{Re\,Y}$ is the conductance

$\mathrm{B} = \mathrm{Im Y}$ is the susceptance

$$\text{real}: G = \frac{R}{R^2 + X^2} \qquad \text{imaginary}: \mathrm{B} = -\frac{X}{R^2 + X^2}$$

Question

Find v(t) and i(t) in the circuit shown in Fig



Kirchhoff's Laws in the frequency Domain

In the sinusoidal steady state , each voltage may be written in cosine form

$$V_{m1}\cos(\omega t+\theta_1)+V_{m2}\cos(\omega t+\theta_2)+\ldots+V_{mn}\cos(\omega t+\theta_n)=0$$

$$\mathrm{Re}(V_{m1}e^{j\theta_1}e^{j\omega t})+\mathrm{Re}(V_{m2}e^{j\theta_2}e^{j\omega t})+\ldots+\mathrm{Re}(V_{mn}e^{j\theta n}e^{j\omega t})=0$$

$$\mathrm{Re}[(V_{m1}e^{j\theta_1}+V_{m2}e^{j\theta_2}+\ldots+V_{mn}e^{j\theta n})e^{j\omega t}]=0$$

If we let $\mathrm{V_k}=V_{mk}e^{j\theta_k}$ then

$$\mathrm{Re}[(\mathrm{V_1}+\mathrm{V_2}+\ldots+\mathrm{V_n})\mathrm{e}^{j\omega t}]=0$$

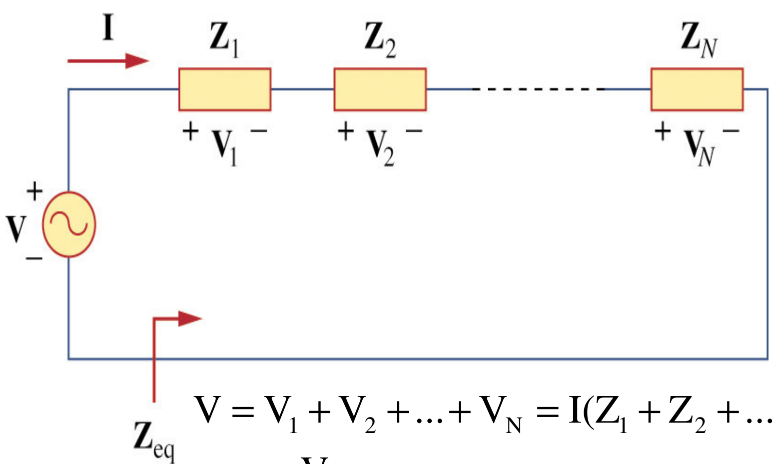
since $\mathrm{e}^{j\omega t}\neq 0$

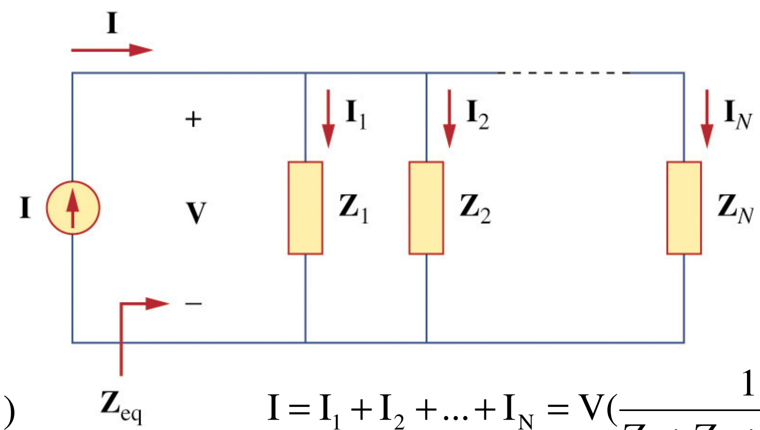
$$\mathrm{V_1}+\mathrm{V_2}+\ldots+\mathrm{V_n}=0$$

By following a similar procedure, we can show that KCL holds for phasor

$$\mathrm{I_1}+\mathrm{I_2}+\ldots+\mathrm{I}_n=0$$

Impedance Combinations





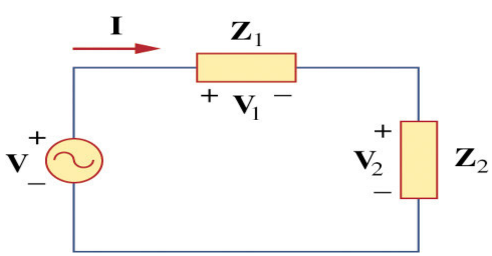
$$V = V_1 + V_2 + \ldots + V_N = I(Z_1 + Z_2 + \ldots + Z_N)$$

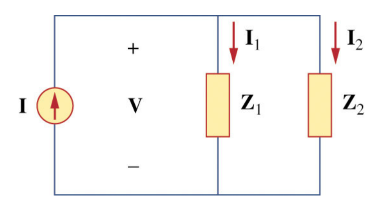
$$Zeq = \frac{V}{I} = Z_1 + Z_2 + \ldots + Z_3$$

$$I = I_1 + I_2 + \ldots + I_N = V(\frac{1}{Z_1 + Z_2 + \ldots + Z_N})$$

$$\frac{1}{Zeq} = \frac{I}{V} = \frac{1}{Z_1} + \frac{1}{Z_2} + \ldots + \frac{1}{Z_3}$$

$$Yeq = Y_1 + Y_2 + \ldots + Y_N$$





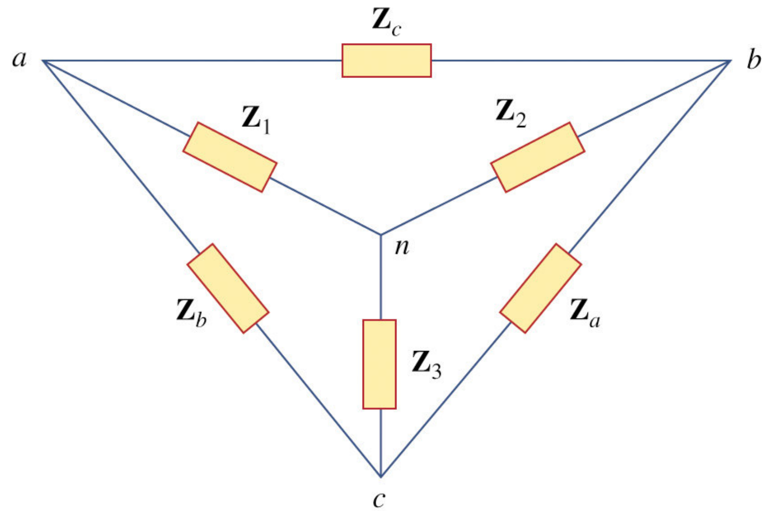
If $N = 2$, the current through the impedance is $I = \dfrac{V}{Z_1 + Z_2}$

If $N = 2$, the *e*quivalent impedance becomes

$$Zeq = \frac{1}{Yeq} = \frac{1}{Y_1 + Y_2} = \frac{1}{1/Z_1 + 1/Z_2} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

$$V = IZeq = I_1 Z_1 = I_2 Z_2$$

Delta-to-wye and Wye-to-delta



Y to δ

$$Z_a = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1}$$

$$Z_b = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2}$$

$$Z_c = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}$$

δ to Y

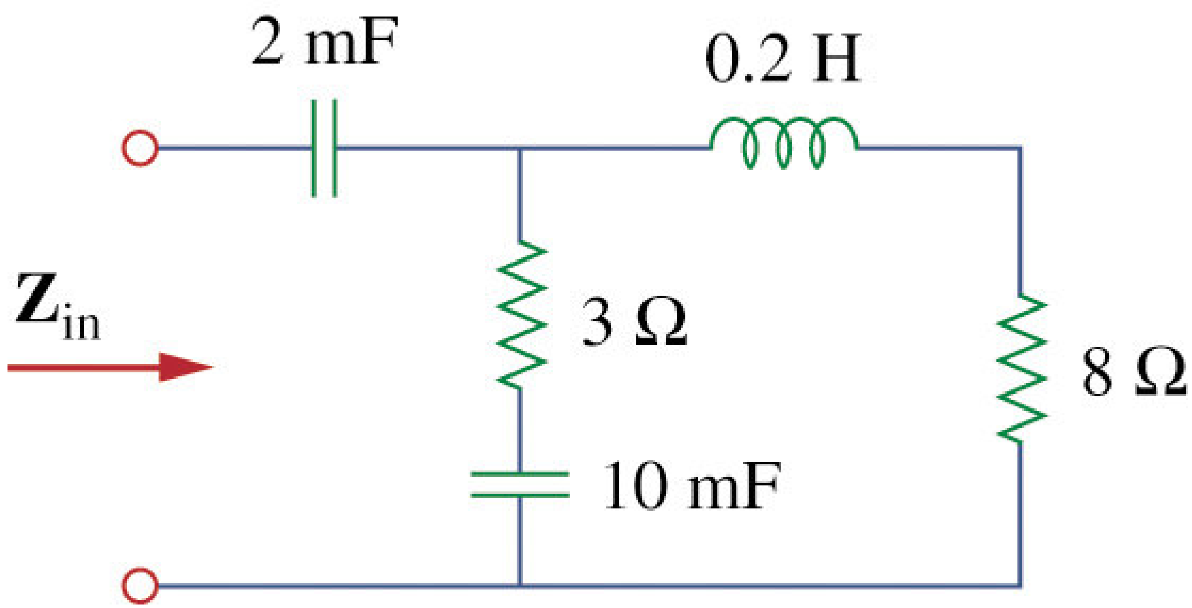
$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

Question

Find the input impedance of the circuit in below Fig. Assume that the circuit operates at $\omega = 50$ rad/s



AC Circuit Sinusiods and Phasors

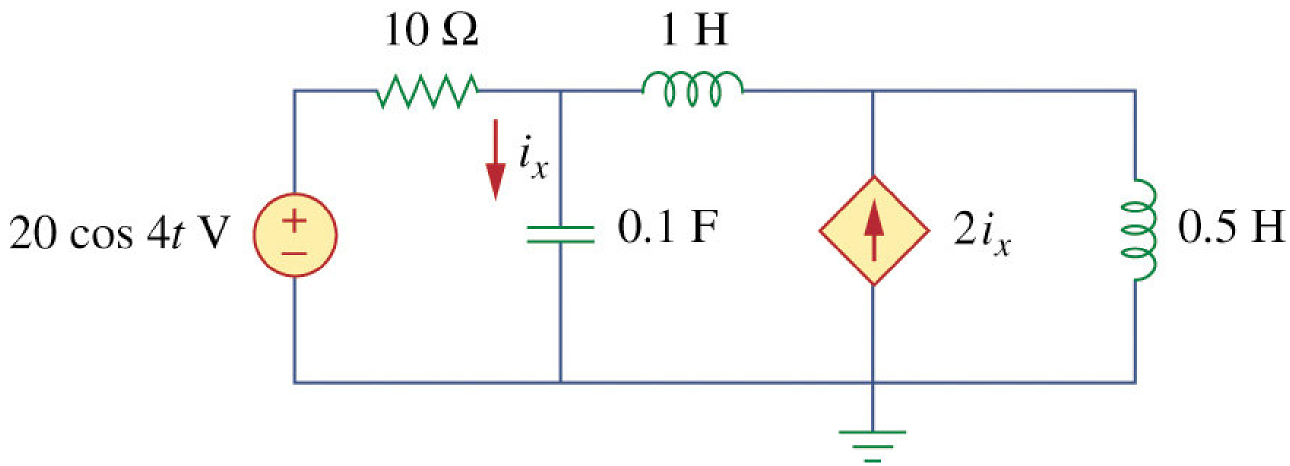
Introduction

Analyzing ac circuits usually requires three steps:

1. Transform the circuit to the phasor or frequency domain " This is not necessary if the problem is specified in the frequency domain"
2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc) "same as dc circuit analysis except that complex numbers are involved"
3. Transform the resulting phasor to the time domain

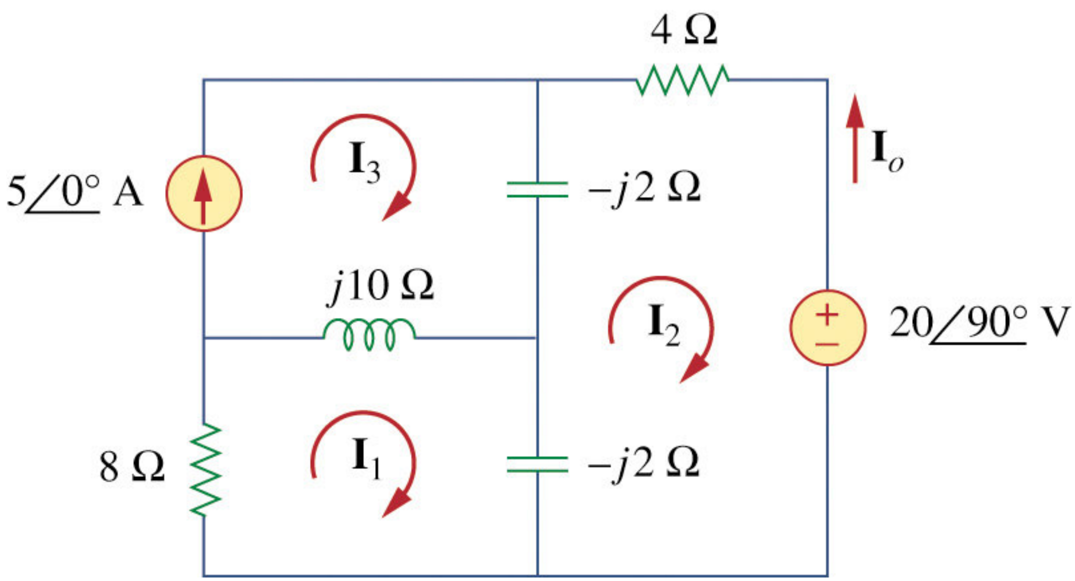
Nodal Analysis

Find i_x in the circuit using nodal analysis



Mesh Analysis

Determine current I_0 in the circuit using mesh analysis.

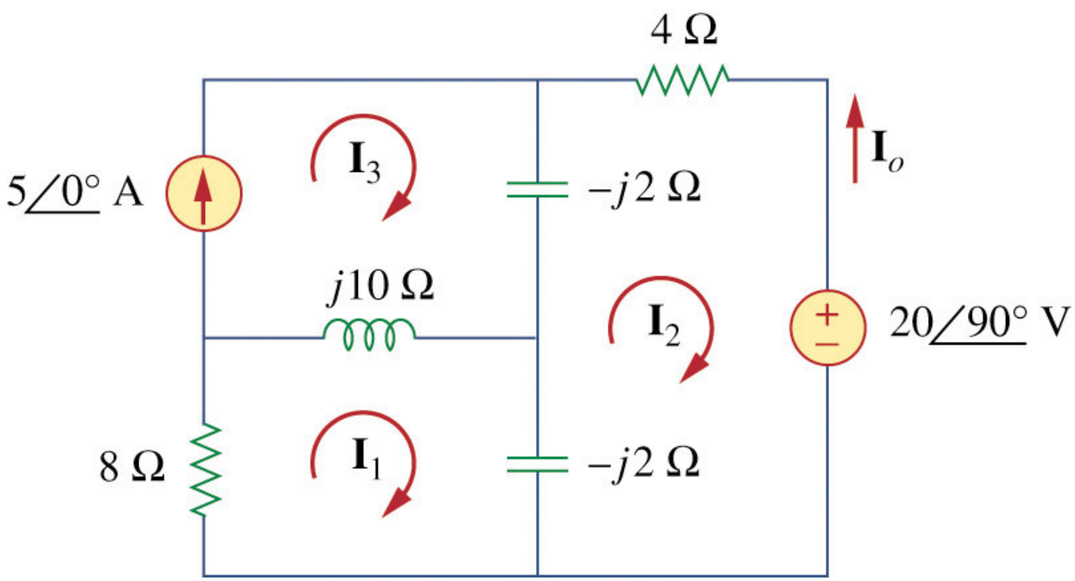


Superposition Theorem

Superposition theorem applies to ac circuit the same way it applies to dc circuits. The theorem becomes important if the circuit has sources operating at different frequencies. Since the impedance (z) depend on frequency, there must be a different frequency domain for each frequency. The total response must be obtained by adding the individual responses in the time domain.

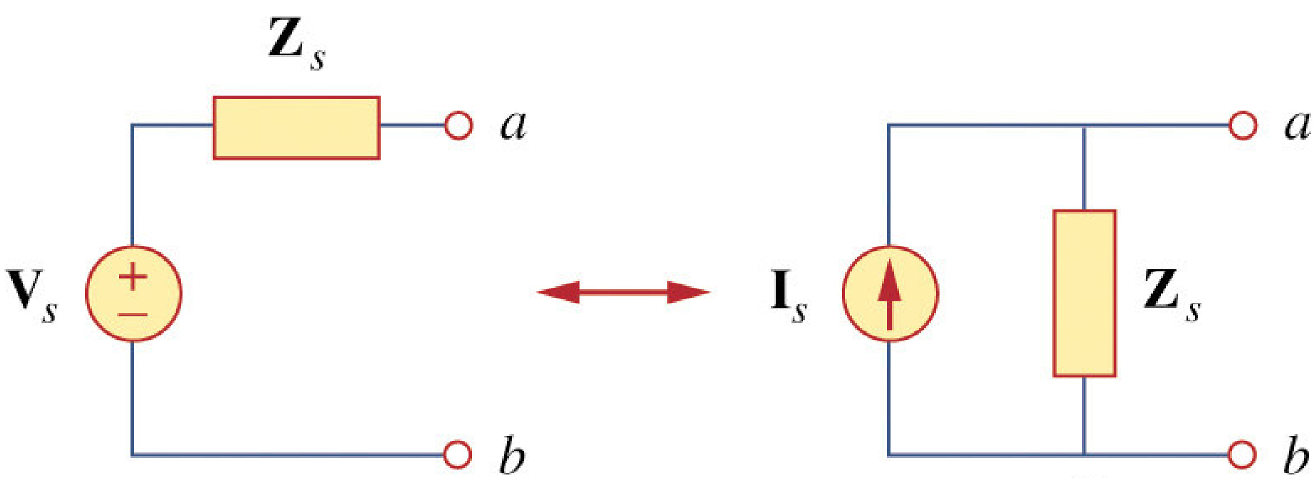
Superposition Theorem

Use the superposition theorem to find I_0 in the circuit



Source Transformation

$$\mathrm{V}_s = \mathrm{Z}_s \mathrm{I}_s \quad \Leftrightarrow \quad \mathrm{I}_s = \frac{\mathrm{V}_s}{\mathrm{Z}_s}$$

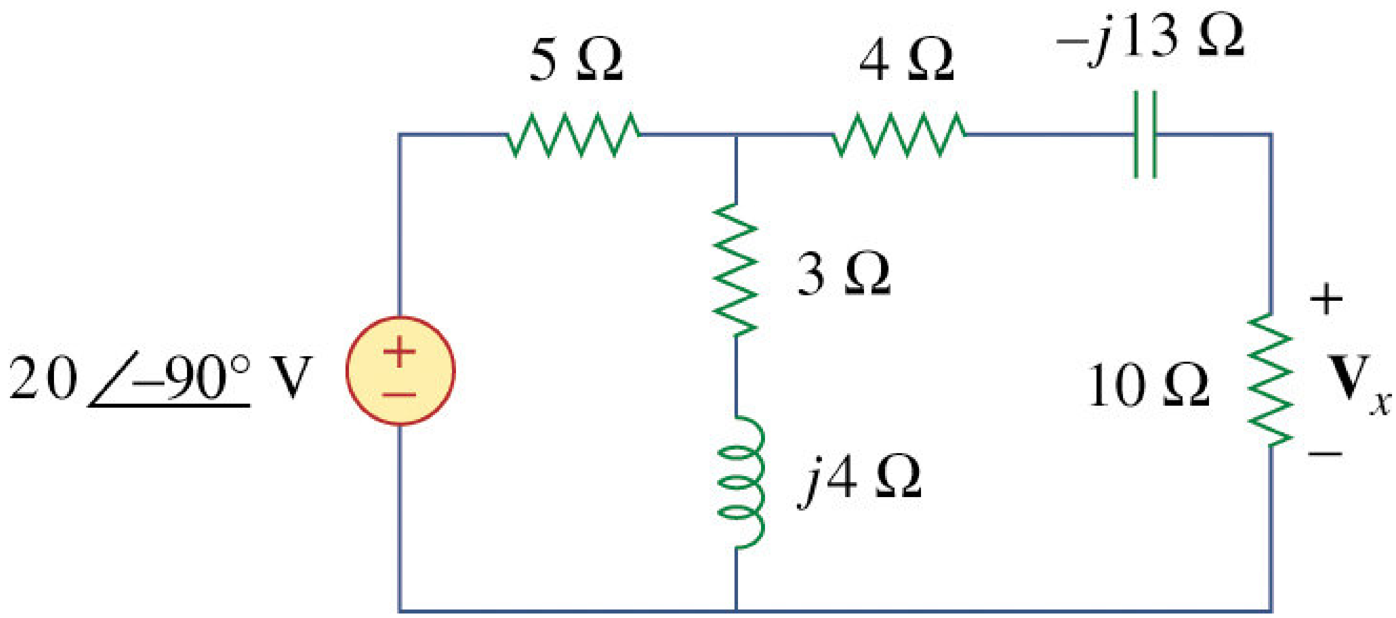


$\mathbf{V}_s = \mathbf{Z}_s \mathbf{I}_s$

$\mathbf{I}_s = \dfrac{\mathbf{V}_s}{\mathbf{Z}_s}$

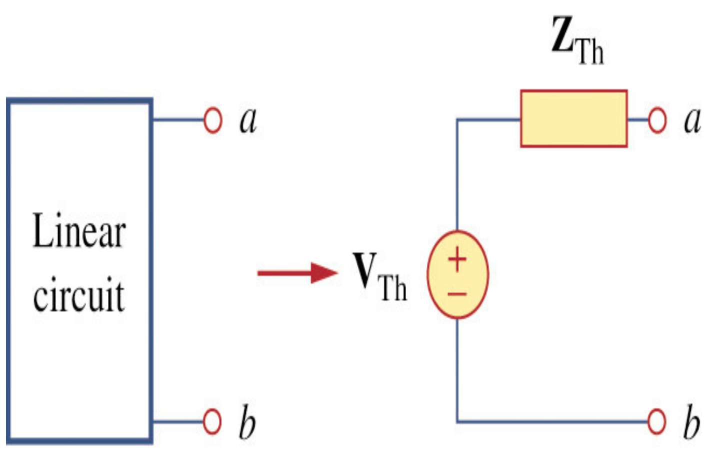
Source Transformation

Calculate V_x in the circuit using the method of source transformation

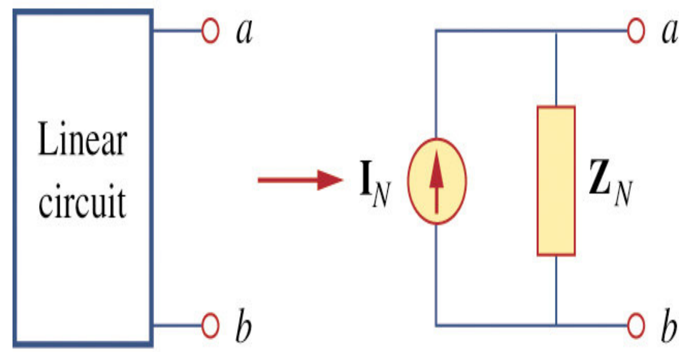


Thevenin and Norton Equivalent Circuits

$$V_{Th} = Z_N I_N \qquad Z_{Th} = Z_N$$



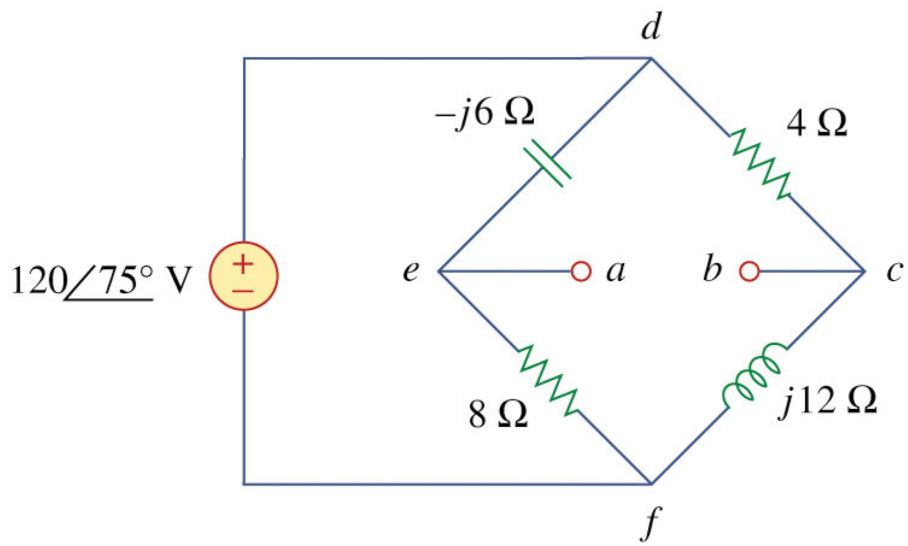
Thevenin Equivalent



Norton Equivalent

Thevenin and Norton Equivalent Circuits

Obtain the Thevenin equivalent at terminal a-b of the circuit



Frequency Response

Introduction

Until now we have learned how to find voltages and currents with a constant frequency (ω). If we let the amplitude of the sinusoidal source remain constant and vary the frequency, we obtain the circuit's frequency response.

The sinusoidal steady-state frequency responses of circuits are of significance in many applications, especially in communications and control system. A specific application is in electrical filters that block out or eliminate signals with unwanted frequencies and pass signals of the desired frequency.

Transfer Function

Transfer function $H(\omega)$ (network function) is a useful analytical tool for finding the frequency response of a circuit.

A transfer function is the frequency-dependent ratio of a forced function to a forcing function (or of an output to an input).

The transfer function $H(\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $Y(\omega)$ (an element voltage or current) to a phasor input $X(\omega)$(source voltage or current).

Thus,

$$\mathrm{H}(\omega) = \frac{\mathrm{Y}(\omega)}{\mathrm{X}(\omega)}$$

There are four possible transfer functions:

$$H(\omega) = \text{Voltage gain} = \frac{V_0(\omega)}{V_i(\omega)}$$

$$H(\omega) = \text{Current gain} = \frac{I_0(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer Impedance} = \frac{V_0(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer Admittance} = \frac{I_0(\omega)}{V_i(\omega)}$$

Cont…

The transfer function $H(\omega)$ can be expressed in terms of its numerator polynomial $N(\omega)$ and denominator polynomial $D(\omega)$ as

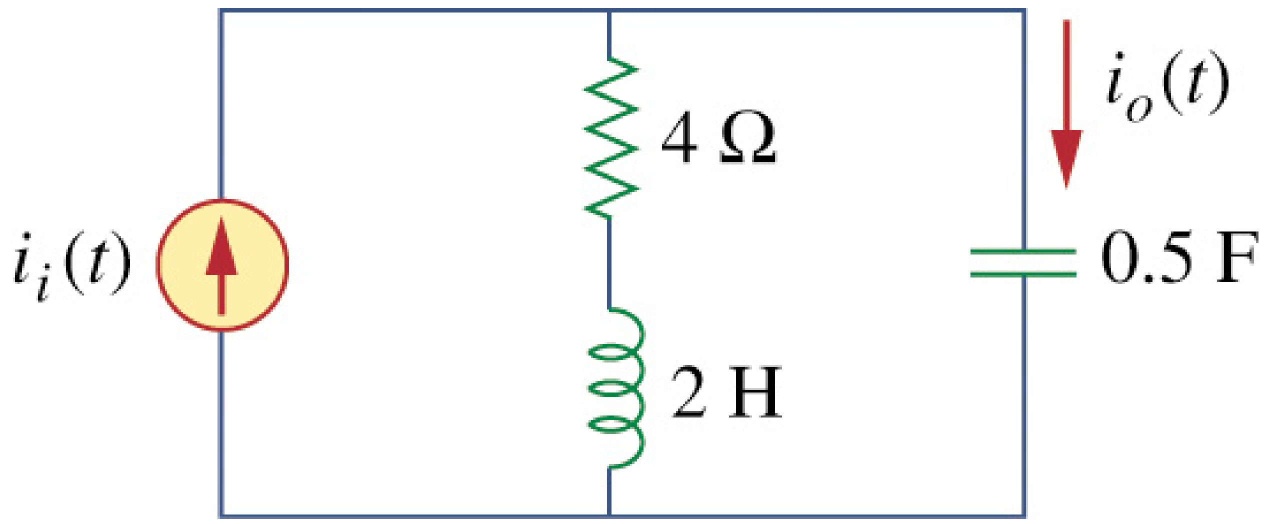
$$H(\omega) = \frac{N(\omega)}{D(\omega)}$$

The roots of $N(\omega)=0$ are called the zeros of $H(\omega)$ and are usually represented as $j\omega=z_1,z_2,\ldots$ Similarly, the roots of $D(\omega)=0$ are the poles of $H(\omega)$ and are represented as $j\omega=p_1,p_2\ldots$

To avoid complex algebra, replace $j\omega$ temporarily with s when working with $H(\omega)$ and replace s with $j\omega$ at the end.

Question 1

For the circuit shown below, calculate the gain $I_0(w)/I_i(w)$ and its poles and zeros.



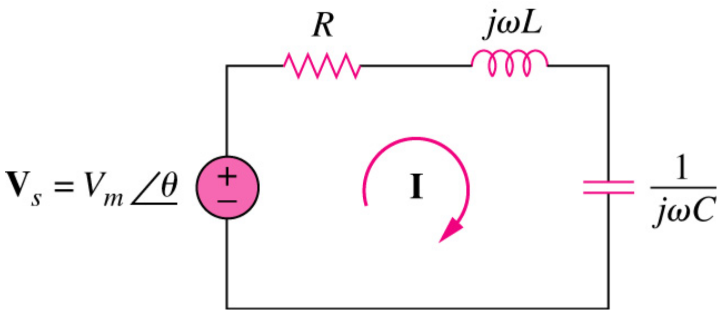
Series Resonance

- Resonance is a condition in an RLC circuit in which the capacitive and inductive reactances are equal in magnitude, thereby resulting in a purely resistive impedance.

- Resonant circuits are useful for constructing filters, as their transfer functions can be highly selective. They are used in many applications such as selecting the desired stations in radio and TV receivers.

Series Resonance

- Consider this series RLC circuit. The input impedance is:



$$Z = R + j\omega L + \frac{1}{j\omega C}$$

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

- Resonance results when the imaginary part of the transfer function is zero.

$$\omega L - \frac{1}{\omega C} = 0$$

Series Resonance

- *The value of ω that satisfies this condition is called the resonant frequency ω_o. Thus, the resonance condition is:*

$$\boxed{\omega_o = \frac{1}{\sqrt{LC}} \, rad/s}$$

- *The average power dissipated by the RLC circuit is:*

$$\boxed{P(\omega) = \frac{1}{2} I^2 R}$$

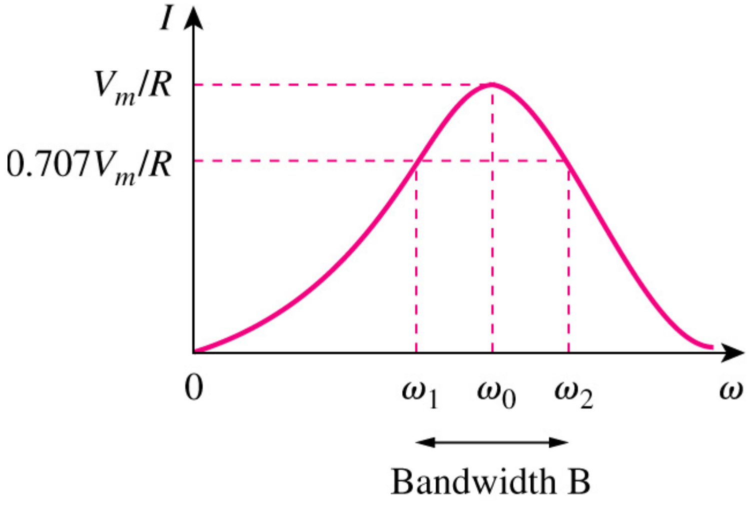
- *The highest power dissipated occurs at resonance, when* $I = \dfrac{V_m^2}{R}$

Series Resonance

- *Half-power frequencies are obtained by setting Z equal to ($\sqrt{2}$)R.*

- *Therefore, the half-power frequencies are:*

$$\boxed{\omega_{1,2} = \mp\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}}$$



- *The width of the response curve depends on the bandwidth B:*

$$\boxed{B = \omega_2 - \omega_1}$$

Series Resonance

- The quality factor (Q) of a resonant circuit is the ratio of its resonant frequency to its bandwidth.

$$\boxed{Q = \frac{\omega_o L}{R} = \frac{1}{\omega_o CR}}$$

- Thus,

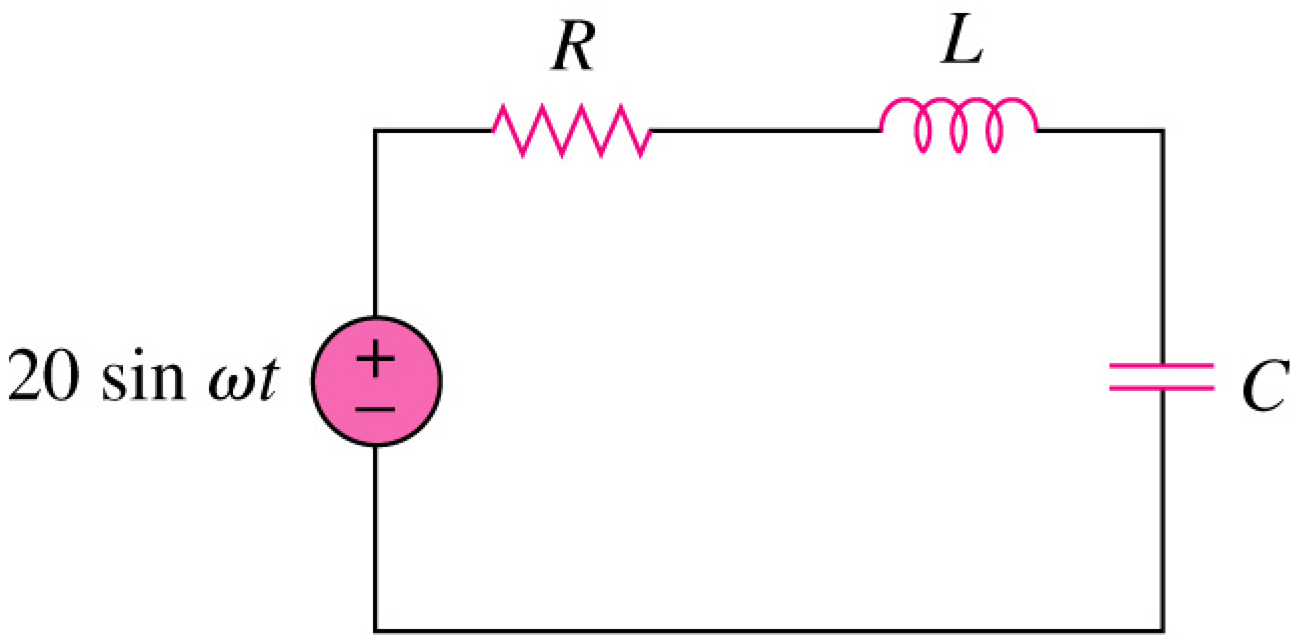
$$\boxed{B = \frac{R}{L} = \frac{\omega_o}{Q}}$$

- *For any circuit with a high Q (Q ≥ 10), the half-frequencies can be found using:

$$\omega_{1,2} = \omega_o \mp \frac{B}{2}$$

Question 2

For the circuit shown below, R = 2 Ω, L =1 mH and C = 0.4 μF. (a) Find the resonant frequency and the half-power frequencies. (b) Calculate the quality factor and bandwidth.



Parallel Resonance

- Resonant frequency remains the same:

$$\omega_o = \frac{1}{\sqrt{LC}} \, rad/s$$

- For half-power frequencies, replace R, L and C in the series circuit with 1/R, C and L respectively.

$$\omega_{1,2} = \mp \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

$$Q = \frac{\omega_o}{B} = \omega_o RC = \frac{R}{\omega_o L}$$

- Again, for Q ≥ 10,

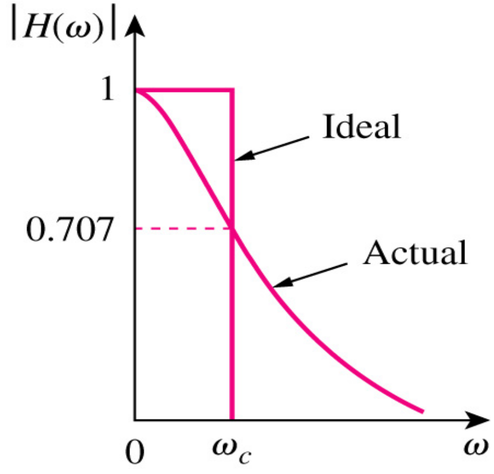
$$\omega_{1,2} = \omega_o \mp \frac{B}{2}$$

Passive Filters

- *A filter is a circuit that is designed to pass signals with desired frequencies and reject or attenuate others.*

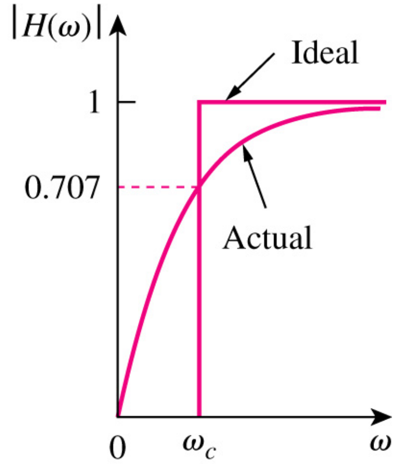
1) Lowpass filter:

- *Designed to pass only frequencies below the cutoff frequency* $\omega_c = \dfrac{1}{RC}$



2) Highpass filter:

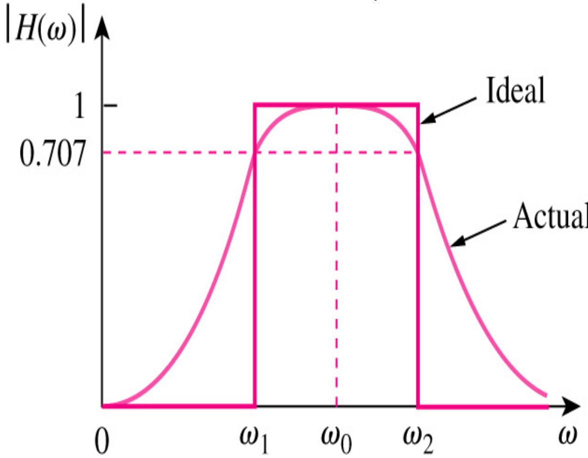
- *Designed to pass only frequencies above the cutoff frequency* $\omega_c = \dfrac{1}{RC}$



Passive Filters

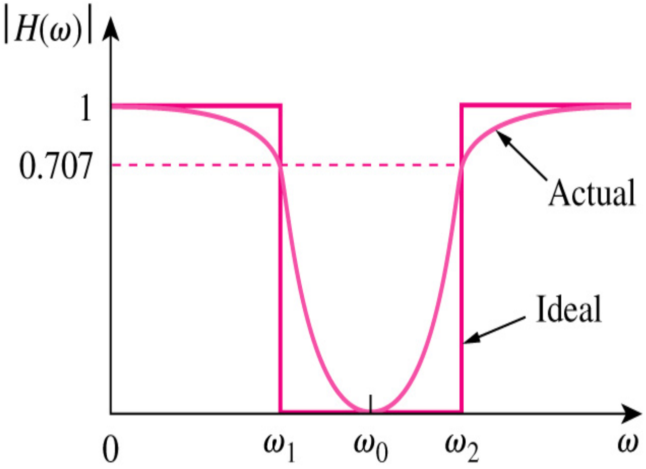
3) Bandpass filter:

- *Designed to pass all frequencies within a band,* $\omega_1 < \omega < \omega_2$



4) Bandstop filter:

- *Designed to stop all frequencies within a band,* $\omega_1 < \omega < \omega_2$



- *For both these cases,* $\omega_o = \dfrac{1}{\sqrt{LC}}$

Active Filters

- *There are 3 major limitations to the passive filters considered in the previous section:*

i) They cannot generate gain greater than 1.

ii) They may require bulky and expensive inductors.

iii) They perform poorly at frequencies below the audio range.

- *Active filters consist of combinations of resistors, capacitors and op-amps.*